Event-by-Event Deep Inelastic Lightcone Kinematics

Charles E. Hyde
Old Dominion University, Norfolk VA 23529

(Dated: 12 September 2016, version 1.0)

For deep inelastic lepton scattering, I define an event-by-event set of Lorentz-covariant basis four-vectors e_X^μ, e_Y^μ, n^μ, and $\widetilde{n}^\mu$. These vectors have the normalization, orthogonality, and handedness properties:

$$e_X^2 = -1 = e_Y^2, \qquad e_X \cdot e_Y = 0 = e_{X,Y} \cdot n = e_{X,Y} \cdot \widetilde{n}, \qquad n^2 = 0 = \widetilde{n}^2, \qquad n \cdot \widetilde{n} = 1$$
$$e_{\mu\nu\alpha\beta}\, n^\mu \widetilde{n}^\nu e_X^\alpha e_Y^\beta = 1$$

In the virtual photon plus incident ion CM frame, the space components of n and $\widetilde{n}$ are anti-parallel and parallel to $\mathbf{P}$ and $\mathbf{q}$, respectively. In this same frame, the time components of e_X and e_Y vanish, the space components of e_X lie in the electron scattering plane, and the space 3-vector of e_Y is perpendicular to this plane.

Jet-clustering algorithms agregate particles in rapidity $\times$ azimuth $\times$ perpendicular momentum, all relative to a preffered axis. For pp or AA scattering, this is simply the beam axis. For DIS, it is preferable to define rapidity η with respect to the $n^\mu, \widetilde{n}^\mu$ vectors defined here (rather than with respect to the detector axis). Once these four basis vectors are constructed, then for an arbitrary final state particle of momentum p^μ in the event:

$$\eta = \frac{1}{2} \ln \left[\frac{\widetilde{n} \cdot p}{n \cdot p} \right], \qquad p_T \cos\phi = -p \cdot e_X, \qquad p_T \sin\phi = -p \cdot e_Y$$

Note that these definitions are Lorentz-invariant and do not require boosting each particle's momentum four-vector to the CM frame.

CONTENTS

I. Collider Kinematics 2
A. Reference Light-Like Four-Vectors 2
B. Defining Rapidity with the Light-Cone Vectors 4
C. Transverse Unit Vectors 4

II. Jet Algorithm Results 6

A. Contractions of the Levi-Civita Symbol 8

B. Alternate LightCone Normalizations 8
1. Unit Normalization 9
2. Beam Normalization 9

References 10

I. COLLIDER KINEMATICS

In these notes, inclusive electron scattering on an ion:

$$e^A Z \to e' X \tag{1}$$

is described by the momentum four-vectors k^μ and k'^μ for the incident and scattered electron, respectively, and P_A^μ for the target ion of mass number A and charge Ze. The virtual photon four-vector is denoted $q^\mu = (k - k')^\mu$. The Bjorken variable is defined with respect to single nucleon kinematics:

$$x_B = \frac{Q^2}{2q \cdot P_A/A} = \frac{Q^2}{2q \cdot P}, \qquad 0 < x_B \leq A \tag{2}$$

with $P = P_A/A = (Z/A)P_0$ the per-nucleon momentum of the beam. Unless otherwise specified, all non-invariant quantities are expressed in the collider (detector) frame.

If the ring magnets are tuned to store protons of momentum P_0, then the stored ion beam will have total momentum

$$P_A = ZP_0 \tag{3}$$

since the magnetic field sets the rigidity (momentum over charge), not the total momentum P_A. The RF frequency and/or the total path length of the lattice has to adjust to the slower ions, as compared to protons, but otherwise the fully stripped ions of momentum ZP_0 behave the same in the magnetic lattice as protons of momentum P_0.

In a proposed electron ion collider, the electron and ion beams collide with a non-zero crossing angle. In particular, in the detector reference frame of the JLab EIC design [1]:

$$\begin{aligned} k^{[0,1,2,3]} &= k\,[1,\,0,\,0,\,-1] \\ P^{[0,1,2,3]} &= \left[\sqrt{\mathbf{P}^2 + M^2},\, |\mathbf{P}|\sin\theta_C\cos\phi_C,\, |\mathbf{P}|\sin\theta_C\sin\phi_C, |\mathbf{P}|\cos\theta_C\right] \end{aligned} \tag{4}$$

with $\theta_C = 0.050$ radians and $\phi_C = \pi$ radians [1]. In the eRHIC design, the crossing angle is smaller, but non-zero. The total invariant CM energy squared is:

$$s_A = (k + P_A)^2 = M_A^2 + 2k \cdot P_A = M_A^2 + 2k\,(E_A + P\cos\theta_C] \tag{5}$$

and the total invariant CM energy squared per nucleon can be defined as:

$$s_N = (k+P)^2 = M_N^2 + 2k \cdot P = M_N^2 + 2k\left(\sqrt{\mathbf{P}^2 + M_N^2} + |\mathbf{P}|\cos\theta_C\right) \tag{6}$$

A. Reference Light-Like Four-Vectors

In the deep inelastic limit of $Q^2 = -q^2$ and $2q \cdot P$ both large ($\gg \Lambda_{\rm QCD}$), but at fixed $x_{\rm B}$, it is useful to define event-by-event light-cone vectors n^μ, $\tilde{n}^\mu$:

$$n \cdot n = 0 = \tilde{n} \cdot \tilde{n} \qquad n \cdot \tilde{n} = 1 \tag{7}$$

such that

$$\begin{aligned} q^\mu &= q^+ n^\mu + q^- \tilde{n}^\mu \qquad q^+ \equiv q\cdot\tilde{n} \qquad q^- \equiv q\cdot n \\ P^\mu &= P^+ n^\mu + P^- \tilde{n}^\mu \qquad P^+ \equiv P\cdot\tilde{n} \qquad P^- \equiv P\cdot n \end{aligned} \tag{8}$$

The two light-cone vectors must each be a linear combination of q^μ and P^μ:

$$\begin{aligned} n^\mu &= \alpha q^\mu + \beta P^\mu \qquad \tilde{n}^\mu = \tilde{\alpha} q^\mu + \tilde{\beta} P^\mu \\ \tilde{n}^2 = 0 = n^2 &= -\alpha^2 Q^2 + (\alpha\beta) 2q\cdot P + \beta^2 M^2 \\ 0 &= -\frac{Q^2}{2q\cdot P}\left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta} + \frac{M^2}{2q\cdot P} = -x_{\rm B}\left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta} + \frac{M^2 x_{\rm B}}{Q^2} \\ \frac{\alpha}{\beta} &= \left[1 \pm \sqrt{1 + 4M^2 x_{\rm B}^2/Q^2}\right] \Big/ (2x_{\rm B}) \end{aligned} \tag{9}$$

Chose the $-$ sign for n^μ and the $+$ sign for $\tilde{n}^\mu$:

$$\begin{aligned} n^\mu &= \beta\left[\frac{q^\mu}{2x_{\rm B}}\left(1 - \sqrt{1+\delta}\right) + P^\mu\right] \qquad \delta \equiv \frac{4x_{\rm B}^2 M^2}{Q^2} \ll 1 \\ \tilde{n}^\mu &= \tilde{\beta}\left[\frac{q^\mu}{2x_{\rm B}}\left(1 + \sqrt{1+\delta}\right) + P^\mu\right] \end{aligned} \tag{10}$$

$$\begin{aligned} 1 = n\cdot\tilde{n} &= \beta\tilde{\beta}\left[\frac{Q^2\delta}{4x_{\rm B}^2} + \frac{q\cdot P}{x_{\rm B}} + M^2\right] = \beta\tilde{\beta}\left[2M^2 + \frac{Q^2}{2x_{\rm B}^2}\right] \\ 1 &= 2M^2\beta\tilde{\beta}\left[1 + \frac{1}{\delta}\right] \quad \Rightarrow \quad \beta\tilde{\beta} = \frac{\delta}{2M^2(1+\delta)} \end{aligned} \tag{11}$$

In the original collider frame, the three-vector components of n and $\tilde{n}$ are not anti-parallel. However, after boosting to the $q + P$ center-of-mass frame, the three vector $\mathbf{n}_{\rm CM}$ is parallel to $\mathbf{P}_{\rm CM}$ and $\tilde{\mathbf{n}}_{\rm CM}$ is parallel to $\mathbf{q}_{\rm CM} = -\mathbf{P}_{\rm CM}$. The choice of the sign from Eq. 9 also ensures that in the Bjorken limit:

$$P^+ q^- \gg -P^- q^+ > 0. \tag{12}$$

with

$$P^+ \equiv \widetilde{n}\cdot P, \qquad P^- \equiv n\cdot P, \qquad etc.. \tag{13}$$

The lightcone vectors n^μ and $\widetilde{n}^\mu$ are defined by Eq. 10, up to a common (though inverse) normalization, provided the product $\beta\tilde{\beta}$ satisfies the constraint of Eq. 11. To make this explicit, define

$$\beta = \frac{\Lambda}{M_N}\sqrt{\frac{\delta}{2(1+\delta)}}, \qquad \tilde{\beta} = \frac{1}{\Lambda M_N}\sqrt{\frac{\delta}{2(1+\delta)}} \tag{14}$$

In the CM frame, a boost parallel or anti-parallel to $\mathbf{P}$ simply changes the value of Λ.

B. Defining Rapidity with the Light-Cone Vectors

In the event-by-event frame defined by the vectors q and P, a generalized definition of rapidity of a particle of momentum p is

$$\eta(p) \equiv \frac{1}{2} \ln \left[\frac{\widetilde{n} \cdot p}{n \cdot p} \right] . \tag{15}$$

This definition is Lorentz-invariant, and hence does not require boosting the vectors to the $q+P$ CM frame. On the other hand, if we change the normalization of the light cone vectors

$$\beta \longrightarrow \Lambda \beta, \qquad \tilde{\beta} \longrightarrow \frac{\tilde{\beta}}{\Lambda} \tag{16}$$

then

$$\eta(p) \longrightarrow \eta(p) - \ln \Lambda \tag{17}$$

The jet-clustering algorithms depend only on the difference $\eta(p_i) - \eta(p_j)$ for all pairs of particles ij. The normalization Λ of the light-cone vectors cancels in this difference.

The momentum fraction of a parton of four-momentum p is defined as

$$x_p = \frac{p \cdot \tilde{n}}{P \cdot \tilde{n}} = \frac{p^+}{P^+} \tag{18}$$

This is obviously independent of the normalization of the light-cone vectors.

Particular choices for the normalization of the light cone vectors are defined in App. B.

C. Transverse Unit Vectors

To analyze the transverse momentum transfer and polarization degrees of freedom in Semi-Inclusive DIS (SIDIS) and Deep Virtual Exclusive Scattering (DVES), and to implement jet-clustering algorithms [2], we require a consistent event-by-event definition of transverse unit vectors. In the target rest frame, the Trento convention [3] is widely used for SIDIS and DVES. In this convention, the instantaneous $\hat{z}$ direction points along $\mathbf{q}_{\rm CM}$. On the other hand, for jet-clustering, and analysis of the parton momentum fractions, it is more convenient to choose $\hat{z}$ along $-\mathbf{q}_{\rm CM} = \mathbf{P}_{\rm CM}$. I introduce a Lorentz-covariant definition of transverse unit vectors, and identify the options for the two conventions.

With $K^\beta = (k + k')^\beta$, define:

$$X^\mu = K^\mu - n^\mu (K \cdot \widetilde{n}) - \widetilde{n}(K \cdot n), \qquad X \cdot n = 0 = X \cdot \widetilde{n} \tag{19}$$

The generalizes the concept of a rest-frame vector that lies in the $k \otimes k'$ scattering plane. The normalization is:

$$\begin{aligned} X^2 &= K^2 - 2(K \cdot n)(K \cdot \widetilde{n}) = K^2 - 2(K \cdot P)^2 \beta \widetilde{\beta} \\ &= -Q^2 \left[\left(\frac{(2-y)}{y} \right)^2 - 1 \right] < 0 \end{aligned} \tag{20}$$

The transverse unit four-vector is

$$e_X^\mu = \frac{X^\mu}{\sqrt{-X^2}} = \frac{K^\mu - n^\mu(K\cdot\widetilde{n}) - \widetilde{n}(K\cdot n)}{(2/y)\sqrt{Q^2\,[1-y]}}, \qquad e_X^2 = -1. \tag{21}$$

Now define a vector orthogonal to e_X, q and P:

$$Y^\mu = g^{\mu\alpha}\epsilon_{\alpha\beta\rho\sigma}P^\beta k^\rho k'^\sigma = \frac{-g^{\mu\alpha}}{2}\epsilon_{\alpha\beta\rho\sigma}P^\beta K^\rho q^\sigma \tag{22}$$

In the Target Rest Frame with a coordinate system such that $k^\mu_{\rm Rest} = k_{\rm Rest}[1,0,0,1]$

$$\begin{aligned}[Y^\mu]_{\rm Rest} &= g^{\mu,2}\epsilon_{2,0,3,1}M\,[kk'\sin\theta_e\cos\phi_e]_{\rm Rest} + g^{\mu,1}\epsilon_{1,0,3,2}M\,[kk'\sin\theta_e\sin\phi_e]_{\rm Rest} \\ &= M\,[0,(\mathbf{k}\times\mathbf{k}')]_{\rm Rest}\end{aligned} \tag{23}$$

Thus Y^μ generalizes the concept of the vector perpendicular to the scattering plane. The normalization of Y^μ is (see App. A):

$$Y^2 = -Q^2\left[(2k\cdot P)(2k'\cdot P) - M^2Q^2\right]/4 = \frac{Q^2(2k\cdot P)^2}{4}\left[1-y-y\frac{\delta}{4}\right] \tag{24}$$

The normalized unit vector is

$$\begin{aligned}e_Y^\mu(\pm) &= \frac{\pm 2g^{\mu\alpha}}{\sqrt{Q^2\,[(2k\cdot P)(2k'\cdot P) - Q^2M^2]}}\epsilon_{\alpha\beta\rho\sigma}P^\beta k^\rho k'^\sigma \\ &= \frac{\mp g^{\mu\mu'}}{\sqrt{Q^2\,[(2k\cdot P)(2k'\cdot P) - M^2Q^2]}}\epsilon_{\mu'\nu\rho\sigma}P^\nu K^\rho q^\sigma\end{aligned} \tag{25}$$

with

$$e_Y^2 = -1 \tag{26}$$

The sign depends on the convention for the coordinate system. In the $q+P$ CM system, if we choose $\hat{z}_q$ along the direction of $\mathbf{q}_{\rm CM}$, then the three space unit-vectors:

$$\mathbf{e}_X^i \quad \otimes \quad \mathbf{e}_Y^j(+) \quad \otimes \quad \hat{z}_q \tag{27}$$

form a right-handed coordinate system consistent with the Trento Convention. This is the preferred choice for studies of SIDIS, TMDs, and DVES. On the other hand, for jet algorithms and parton hadronization, it is more convenient to have the CM z-axis along $\mathbf{P}_{\rm CM}$. In this case

$$\mathbf{e}_X^i \quad \otimes \quad \mathbf{e}_Y^j(-) \quad \otimes \quad \hat{z}_P = -\hat{z}_q \tag{28}$$

also form a right-handed coordinate system.

Equivalently, start with the alternate definition:

$$\begin{aligned}\tilde{Y}^\mu &= g^{\mu\alpha}\epsilon_{\alpha\beta\rho\sigma}K^\beta n^\rho \tilde{n}^\sigma \\ &= Y^\mu \frac{\delta}{M^2 x_{\rm B}\sqrt{1+\delta}} \\ e_Y^\mu(\pm) &= \frac{\pm g^{\mu\alpha}}{\sqrt{Q^2\,[(2k\cdot P)(2k'\cdot P) - M^2]}}\frac{2M^2x_{\rm B}\sqrt{1+\delta}}{\delta}\epsilon_{\alpha\beta\rho\sigma}K^\beta n^\rho\tilde{n}^\sigma\end{aligned} \tag{29}$$

The Lorentz invariant generalization of a 'right-handed' coordinate system can be expressed as

$$\epsilon_{\alpha\beta\rho\sigma}n^\alpha e_X^\beta e_Y^\rho \tilde{n}^\sigma = 1 \qquad \text{with} \quad e_Y = e_Y(+) \tag{30}$$

II. JET ALGORITHM RESULTS

I have implemented the FastJet suite of jet algorithms [2] together with an event analyzer for PYTHIA6 event files of open charm production created by Y. Furletova. I have used the Anti-k_T algorithm with distance measure $R = 2.0$ fm and $p_{\mathrm{T,\ min}} = 0.05$ GeV/c. I am still checking my sign conventions and other details.

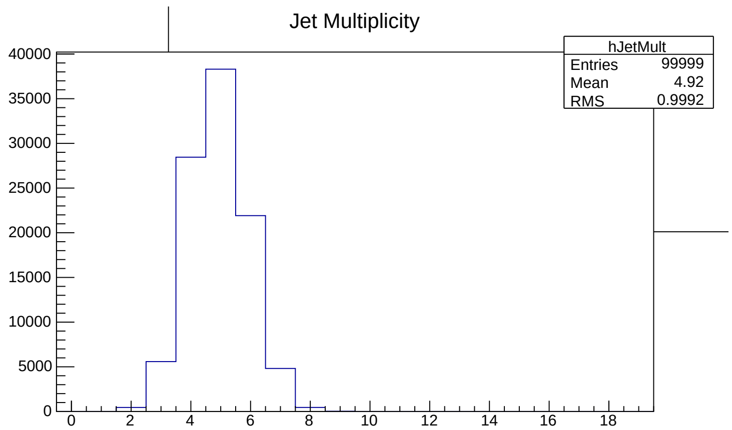

FIG. 1. Jet Multiplicity (number of jets) distribution with the anti-k_T jet algorithm: $R = 2$, $P_T > 0.05$ GeV.

The jet multiplicity for 100,000 Photon Gluon Fusion (PGF) events on the proton ($10 \otimes 100$ GeV^2) is shown in Fig. 1. Assuming the scattered electron is being identified as a jet, then we expect a minimum of 4 jets: The electron, the two charm-quark jets, and the target remnant jet. In principle, every stable final state particle is either uniquely assigned to a jet, or is identified as a jet unto itself (*e.g.* the scattered electron).

A sample single event display is shown in Fig. 2.

I still have checks to do on my code, and intend to develop a single event display showing all particles clustered into jets. One of my goals with this study is to see if the combinatorial background for D-meson reconstruction via charged particle decay modes is reduced if I require all the particles to be found in a single jet.

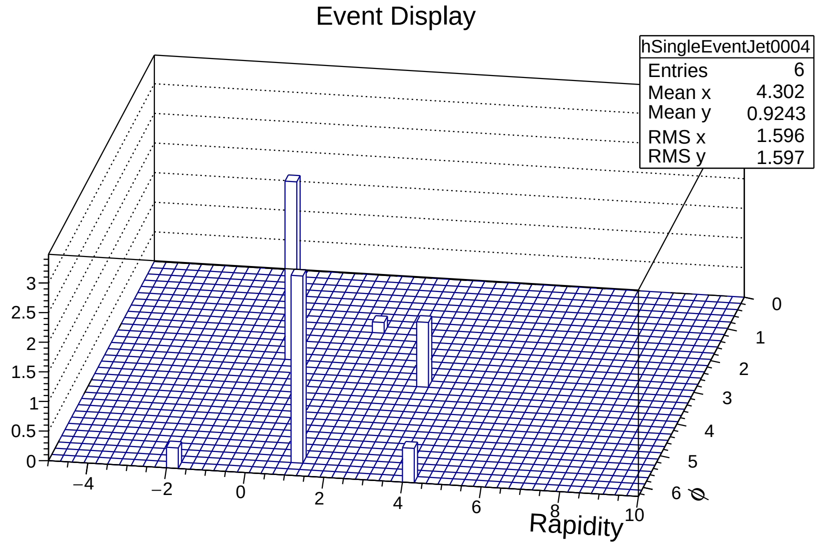

FIG. 2. Sample event with 6 reconstructed jets. Rapidity (η) and ϕ are as defined in the text. The vertical axis is the total P_T of the jet. I have not verified, but I think the low-p_T jets at $\eta \approx -2$ and $\eta \approx 4$ are the scattered electron and the target fragmentation jet, respectively. The two large p_T jets separated by $\Delta\phi \approx \pi$ should be the charm and anti-charm jets.

Appendix A: Contractions of the Levi-Civita Symbol

The fully antisymmetric Levi-Civita symbol $\epsilon_{\alpha\beta\rho\sigma}$ has normalization

$$\epsilon_{0,1,2,3} = 1 = -\epsilon^{0,1,2,3} \tag{A1}$$

The double contraction of two Levi-Civita symbols is [4]:

$$-\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'} = \delta^{\nu}_{\nu'}\left[\delta^{\rho}_{\rho'}\delta^{\sigma}_{\sigma'} - \delta^{\rho}_{\sigma'}\delta^{\sigma}_{\rho'}\right] - \delta^{\nu}_{\rho'}\left[\delta^{\rho}_{\nu'}\delta^{\sigma}_{\sigma'} - \delta^{\rho}_{\sigma'}\delta^{\sigma}_{\nu'}\right] + \delta^{\nu}_{\sigma'}\left[\delta^{\rho}_{\nu'}\delta^{\sigma}_{\rho'} - \delta^{\rho}_{\rho'}\delta^{\sigma}_{\nu'}\right] \tag{A2}$$

Equivalently:

$$\begin{aligned} -g^{\mu\mu'}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu'\nu'\rho'\sigma'} = {} & g_{\nu\nu'}\left[g_{\rho\rho'}g_{\sigma\sigma'} - g_{\rho\sigma'}g_{\sigma\rho'}\right] - g_{\nu\rho'}\left[g_{\rho\nu'}g_{\sigma\sigma'} - g_{\rho\sigma'}g_{\sigma\nu'}\right] \\ & + g_{\nu\sigma'}\left[g_{\rho\nu'}g_{\sigma\rho'} - g_{\rho\rho'}g_{\sigma\nu'}\right] \end{aligned} \tag{A3}$$

Define the vector

$$\widetilde{A}^{\mu} = g^{\mu\mu'}\epsilon_{\mu'\nu\rho\sigma}B^{\nu}C^{\rho}D^{\sigma} \tag{A4}$$

The norm of this vector is

$$\begin{aligned} \widetilde{A}^2 &= g^{\mu\mu'}\left(\epsilon_{\mu\nu\rho\sigma}B^{\nu}C^{\rho}D^{\sigma}\right)\left(\epsilon_{\mu'\nu'\rho'\sigma'}B^{\nu'}C^{\rho'}D^{\sigma'}\right) \\ &= -\Big\{B^2\left[C^2D^2 - (C\cdot D)^2\right] - (B\cdot C)\left[(B\cdot C)D^2 - (C\cdot D)(B\cdot D)\right] \\ &\quad + (B\cdot D)\left[(B\cdot C)(C\cdot D) - C^2(B\cdot D)\right]\Big\} \\ &= -B^2C^2D^2 + B^2(C\cdot D)^2 + C^2(B\cdot D)^2 + D^2(B\cdot C)^2 - 2(B\cdot C)(C\cdot D)(D\cdot B) \end{aligned} \tag{A5}$$

For

$$Y^{\mu} = -g^{\mu\mu'}\epsilon_{\mu'\nu\rho\sigma}P^{\nu}K^{\rho}q^{\sigma}, \tag{A6}$$

we have

$$Y^2 = -Q^2\left[(2k\cdot P)(2k'\cdot P) - M^2\right] \tag{A7}$$

The normalized unit vector, normal to the electron scattering plane is

$$e^{\mu}_{Y} = \frac{Y^{\mu}}{\sqrt{-Y^2}} = \frac{-g^{\mu\mu'}\epsilon_{\mu'\nu\rho\sigma}P^{\nu}K^{\rho}q^{\sigma}}{\sqrt{Q^2\left[(2k\cdot P)(2k'\cdot P) - M^2Q^2\right]}}, \qquad e^2_Y = -1. \tag{A8}$$

Appendix B: Alternate LightCone Normalizations

What is an appropriate normalization Λ of the light-cone vectors n and $\widetilde{n}$, and what is the impact of a different normalization choice?

Section I B has already established that the normalization is arbitrary. In this appendix I offer two choices of normalization that may be useful.

1. Unit Normalization

It will sometimes be convenient to define the normalizations β, $\tilde{\beta}$ such that in the $q+P$ center-of-mass frame, the time-components

$$n^0_{\mathrm{CM}} = \tilde{n}^0_{\mathrm{CM}} = 1/\sqrt{2}. \tag{B1}$$

Then the light-like vectors will have the form:

$$n_{\mathrm{CM}}^{[0,1,2,3]} = \frac{1}{\sqrt{2}}\left[1, \hat{\mathbf{n}}\right] \qquad \tilde{n}_{\mathrm{CM}}^{[0,1,2,3]} = \frac{1}{\sqrt{2}}\left[1, -\hat{\mathbf{n}}\right]. \tag{B2}$$

With this normalization, the rapidity of a particle of 4-momentum p, as defined in § I B has the form

$$\eta(p) = \frac{1}{2}\ln\left[\frac{\widetilde{n}\cdot p}{n\cdot p}\right] \longrightarrow \frac{1}{2}\ln\left[\frac{p^0+p^z}{p^0-p^z}\right]_{\mathrm{CM}} \tag{B3}$$

with the z-axis parallel to $\mathbf{P}_{\mathrm{CM}}$.

In the CM frame:

$$q^0_{\mathrm{CM}} = \frac{W^2-M^2-Q^2}{2W} \qquad\qquad P^0_{\mathrm{CM}} = \frac{W^2+M^2+Q^2}{2W} \tag{B4}$$

Therefore, this normalization choice is

$$\begin{aligned} 1 = n^0_{\mathrm{CM}} &= \beta\left[\frac{q^0_{\mathrm{CM}}}{2x_{\mathrm{B}}}\left(1-\sqrt{1+\delta}\right) + P^0_{\mathrm{CM}}\right] \\ &= \beta\left[\left(\frac{W}{2}\right)\left(\frac{1}{2x_{\mathrm{B}}}+1\right) + \frac{M^2+Q^2}{2W}\left(1-\frac{\sqrt{1+\delta}}{2x_{\mathrm{B}}}\right)\right] \end{aligned} \tag{B5}$$

leading to

$$\begin{aligned} \beta &= \left[\left(\frac{W}{2}\right)\left(\frac{1}{2x_{\mathrm{B}}}+1\right) + \frac{M^2+Q^2}{2W}\left(1-\frac{\sqrt{1+\delta}}{2x_{\mathrm{B}}}\right)\right]^{-1} \\ \widetilde{\beta} &= \frac{\delta}{2M^2(1+\delta)}\left[\left(\frac{W}{2}\right)\left(\frac{1}{2x_{\mathrm{B}}}+1\right) + \frac{M^2+Q^2}{2W}\left(1-\frac{\sqrt{1+\delta}}{2x_{\mathrm{B}}}\right)\right] \end{aligned} \tag{B6}$$

2. Beam Normalization

The per-nucleon rapidity of the incident ion beam in the collider frame, with a z-axis parallel to the beam momentum is:

$$\eta_0 = \frac{1}{2}\ln\left[\frac{E+|\mathbf{P}|}{E-|\mathbf{P}|}\right]_{\mathrm{Detector}} = \ln\left[\frac{E+|\mathbf{P}|}{M_N}\right], \qquad E \equiv \sqrt{M_N^2+P^2} \tag{B7}$$

For ions with $A>1$, the per nucleon energy E is ambiguous, but this is a minor point.

One 'natural' choice of normalization Λ is such that $\eta(P) = \eta_0$:

$$\ln\left[\frac{E+P}{M_N}\right] = \frac{1}{2}\ln\left[\frac{\widetilde{n}\cdot P}{n\cdot P}\right] = \frac{1}{2}\ln\left[\frac{\frac{(q\cdot P)^2}{Q^2}\left(1+\sqrt{1+\delta}\right)+M^2}{\Lambda^2\left(\frac{(q\cdot P)^2}{Q^2}\left(1-\sqrt{1+\delta}\right)+M^2\right)}\right]$$

$$\left[\frac{E+P}{M_N}\right]^2 = \frac{1}{\Lambda^2}\frac{\left(1+\sqrt{1+\delta}\right)+\delta}{\left(1-\sqrt{1+\delta}\right)+\delta}. \tag{B8}$$

Therefore

$$\Lambda = \frac{M_N}{E+P}\sqrt{\frac{\left(1+\sqrt{1+\delta}\right)+\delta}{\left(1-\sqrt{1+\delta}\right)+\delta}} \approx \frac{\sqrt{Q^2}}{(E+P)x_{\mathrm{B}}} \tag{B9}$$

and

$$\beta = \frac{1}{E+|\mathbf{P}|}\sqrt{\frac{\delta}{2(1+\delta)}}\sqrt{\frac{\left(1+\sqrt{1+\delta}\right)+\delta}{\left(1-\sqrt{1+\delta}\right)+\delta}}$$

$$\widetilde{\beta} = \frac{1}{E-|\mathbf{P}|}\sqrt{\frac{\delta}{2(1+\delta)}}\sqrt{\frac{\left(1-\sqrt{1+\delta}\right)+\delta}{\left(1+\sqrt{1+\delta}\right)+\delta}} \tag{B10}$$

with E and $|\mathbf{P}|$ in the detector frame.

---

[1] S. Abeyratne *et al.*, "MEIC Design Summary," arXiv:1504.07961 [physics.acc-ph].

[2] M. Cacciari, G. P. Salam and G. Soyez, Eur. Phys. J. C **72**, 1896 (2012) doi:10.1140/epjc/s10052-012-1896-2 [arXiv:1111.6097 [hep-ph]], also `http://fastjet.fr`

[3] A. Bacchetta, U. D'Alesio, M. Diehl and C. A. Miller, "Single-spin asymmetries: The Trento conventions," Phys. Rev. D **70**, 117504 (2004) doi:10.1103/PhysRevD.70.117504 [hep-ph/0410050].

[4] `http://demonstrations.wolfram.com/ProductOfTwoLeviCivitaTensorsWithContractions/`