Error Analysis

Analytical

02 October 2013

Penfold-Leiss Cross Section Unfolding

- Measure Yields at: $E = E_1, E_2, \ldots, E_n$ where,

$$E_i - E_{i-1} = \Delta, i = 2, n$$

$$Y(E_i) = \int_{th}^{E_i} n_\gamma(E_i, k)\sigma(k)dk \approx \sum_{j=1}^{i} N_\gamma(E_i, \Delta, E_j)\sigma(E_j)$$

- The solution can be written in two forms:

$$\sigma_i = \frac{1}{N_{ii}}\left[y_i - \sum_{j=1}^{i-1}\left(N_{ij}\sigma_j\right)\right]$$

- Or, Matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} N_{\gamma,11} & 0 & \cdots & 0 \\ N_{\gamma,21} & N_{\gamma,22} & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ N_{\gamma,n1} & N_{\gamma,n2} & \cdots & N_{\gamma,nn} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix}$$

$$[Y] = [N] \bullet [\sigma]$$

$$[\sigma] = [N]^{-1} \bullet [Y]$$

Statistical Errors

Statistical Error Propagation (1)

- Note: $$\frac{dy_i}{y_i} = \frac{1}{\sqrt{y_i}} \qquad \frac{dN_{ij}}{N_{ij}} = \frac{1}{\sqrt{N_{ij}}} \approx 0$$

$$dy_i = \sqrt{y_i} \qquad dy_i = \sqrt{y_i + 2y_i^{bg}}$$

In case of background Subtraction

- With:

$$[B] = [N]^{-1}$$

$$[\sigma] = [B] \bullet [Y]$$

- Then:

$$[d\sigma^2] = [B] \bullet [dY^2] \bullet [B]^T$$

- Where:

$$\left[dY^2\right] = \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{bmatrix}$$

$$\mathrm{var}(y_i, y_i) = y_i$$
$$\mathrm{cov}(y_i, y_j) = 0$$

$$\left[d\sigma^2\right] = \begin{bmatrix} d\sigma_1^2 & \mathrm{cov}(\sigma_1, \sigma_2) & \cdots & \mathrm{cov}(\sigma_1, \sigma_n) \\ \mathrm{cov}(\sigma_2, \sigma_1) & d\sigma_2^2 & \cdots & \mathrm{cov}(\sigma_2, \sigma_n) \\ \vdots & \ddots & \ddots & \vdots \\ \mathrm{cov}(\sigma_n, \sigma_1) & \mathrm{cov}(\sigma_n, \sigma_2) & \cdots & d\sigma_n^2 \end{bmatrix}$$

Statistical Error Propagation (2)

$$(d\sigma_i)^2 = \frac{1}{N_{ii}^2}\left[dy_i^2 + \sum_{j=1}^{i-1}\left(N_{ij}d\sigma_j\right)^2 + \sum_{k=1}^{i-1}\sum_{l=1}^{i-1} N_{ik}\,\mathrm{cov}(\sigma_k,\sigma_l)N_{il} \right]$$

Although,

$$\mathrm{cov}(y_i, y_j) = 0,$$

$$\mathrm{cov}(\sigma_i, \sigma_j) \neq 0$$

For mono-chromatic beam

$$\left(\frac{d\sigma_i}{\sigma_i}\right)^2 = \left(\frac{dy_i}{y_i}\right)^2 = \frac{1}{y_i}$$

Statistical Error Propagation (Wrong)

$$[\sigma] = [N]^{-1} \bullet [Y]$$

- Then:

Wrong

$$\left[d\sigma^2\right] = \left[N^2\right]^{-1} \bullet \left[dY^2\right]$$

- This is equivalent to:

$$(d\sigma_i)^2 = \frac{1}{N_{ii}^2}\left[dy_i^2 - \sum_{j=1}^{i-1}\left(N_{ij}\, d\sigma_j\right)^2\right]$$

Wrong

Absolute Systematic Errors

Systematic Error Propagation (1)

- For absolute beam energy uncertainty of δE (= 0.1%):

$$\frac{dy_i}{y_i} = \frac{y_i(E_i + \delta E) - y_i(E_i)}{y_i(E_i)} \qquad \frac{dN_{ij}}{N_{ij}} = \frac{N_{ij}(E_i + \delta E) - N_{ij}(E_i)}{N_{ij}(E_i)}$$

E_i (MeV)	dy_i/y_i (%)	$d\sigma_i/\sigma_i$ (%)
7.9	17.4	12.6
8.0	12.3	10.5
8.1	10.0	9.1
8.2	8.6	7.1
8.3	7.6	6.3
8.4	6.8	5.8
8.5	6.1	5.2

This is the cross section dependence on energy

- Accounted for dN_{ij} due to energy error when calculating dy_i

$$\approx \frac{\delta E}{i\Delta}$$

$$\left[dN_{ij}/N_{ij}\right] = \begin{bmatrix} 0.100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.058 & 0.050 & 0 & 0 & 0 & 0 & 0 \\ 0.041 & 0.039 & 0.033 & 0 & 0 & 0 & 0 \\ 0.031 & 0.031 & 0.029 & 0.025 & 0 & 0 & 0 \\ 0.025 & 0.025 & 0.025 & 0.023 & 0.020 & 0 & 0 \\ 0.021 & 0.021 & 0.021 & 0.021 & 0.020 & 0.017 & 0 \\ 0.018 & 0.018 & 0.018 & 0.018 & 0.018 & 0.017 & 0.022 \end{bmatrix}$$

- With:

$$[B] = [N]^{-1}$$

$$[\sigma] = [B] \bullet [Y]$$

- Then:

$$\left[d\sigma^2\right] = [B] \bullet \left(\left[dY^2\right] + \left[dN^2\right] \bullet \left[\sigma^2\right]\right) \bullet [B]^T$$

- Where:

$$[dN^2] = \begin{bmatrix} (dN_{11})^2 & 0 & \cdots & 0 \\ (dN_{21})^2 & (dN_{22})^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (dN_{n1})^2 & (dN_{n2})^2 & \cdots & (dN_{nn})^2 \end{bmatrix}$$

$$[dY^2] = \begin{bmatrix} (dy_1)^2 & 0 & \cdots & 0 \\ 0 & (dy_2)^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & (dy_n)^2 \end{bmatrix} \qquad [\sigma^2] = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

Systematic Error Propagation (2)

$$\left(d\sigma_i\right)^2 \cong \frac{1}{N_{ii}^2}\left[dy_i^2 + \sum_{j=1}^{i-1}\left(N_{ij}d\sigma_j\right)^2 + \sum_{k=1}^{i-1}\sum_{l=1}^{i-1} N_{ik}\,\mathrm{cov}(\sigma_k,\sigma_l)N_{il}\right.$$
$$\left. + \sum_{j=1}^{i-1}\left(dN_{ij}\sigma_j\right)^2 + \left(dN_{ii}\sigma_i\right)^2\right]$$

Other Absolute Systematic Errors

Beam Current, $\delta I/I$	3%
Photon Flux, $\delta\varphi/\varphi$	5%
Radiator Thickness, $\delta R/R$	3%
Bubble Chamber Thickness, $\delta T/T$	3%
Bubble Chamber Efficiency, ε	5%

Simulation

- Then:

$$(dy_i)^2 = (dy_i(\delta E))^2 + \left[\left(\frac{\delta I}{I}\right)^2 + \left(\frac{\delta R}{R}\right) + \left(\frac{\delta T}{T}\right)^2 + \varepsilon^2\right] y_i^2$$

$$(dN_{ij})^2 = \left(\frac{\delta\phi}{\phi}\right)^2 N_{ij}^2$$

Results

I. Radiator Thickness = 0.02 mm
II. Bubble Chamber Thickness = 3.0 cm
III. Number of ^{16}O nuclei = 3.474e22 /cm^2
IV. Background subtraction of $^{18}O(\gamma,\alpha)^{14}C$
V. $^{17}O(\gamma,n)^{16}O$: Still to do

Electron Beam K. E.	Beam Current (μA)	Time (hour)	y_i	dy_i (no bg)	dy_i/y_i (no bg, %)	dy_i (with bg)	dy_i/y_i (with bg, %)
7.9	100	100	545	23	4.2	134	24.6
8.0	100	20	581	24	4.1	77	13.3
8.1	80	10	852	29	3.4	60	7.0
8.2	20	10	634	25	3.9	40	6.3
8.3	10	10	812	28	3.4	39	4.8
8.4	4	10	746	27	3.6	36	4.8
8.5	2	10	763	28	3.7	32	4.2

$$[N]=\begin{bmatrix} 3.267e14 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9.782e13 & 6.439e13 & 0 & 0 & 0 & 0 & 0 \\ 5.013e13 & 3.858e13 & 2.539e13 & 0 & 0 & 0 & 0 \\ 1.494e13 & 1.236e13 & 9.514e12 & 6.258e12 & 0 & 0 & 0 \\ 8.540e12 & 7.369e12 & 6.097e12 & 4.692e12 & 3.086e12 & 0 & 0 \\ 3.801e12 & 3.370e12 & 2.908e12 & 2.406e12 & 1.852e12 & 1.217e12 & 0 \\ 2.075e12 & 1.875e12 & 1.663e12 & 1.435e12 & 1.187e12 & 9.137e11 & 6.004e11 \end{bmatrix}$$

Electron Beam K. E.	Cross Section (nb)	Stat Error (no bg, %)	Stat Error (with bg, %)
7.9	0.046	4.4	24.5
8.0	0.185	6.0	20.7
8.1	0.58	6.3	14.7
8.2	1.53	8.2	13.8
8.3	3.49	9.1	13.3
8.4	7.2	10.6	13.8
8.5	13.6	12.2	14.8

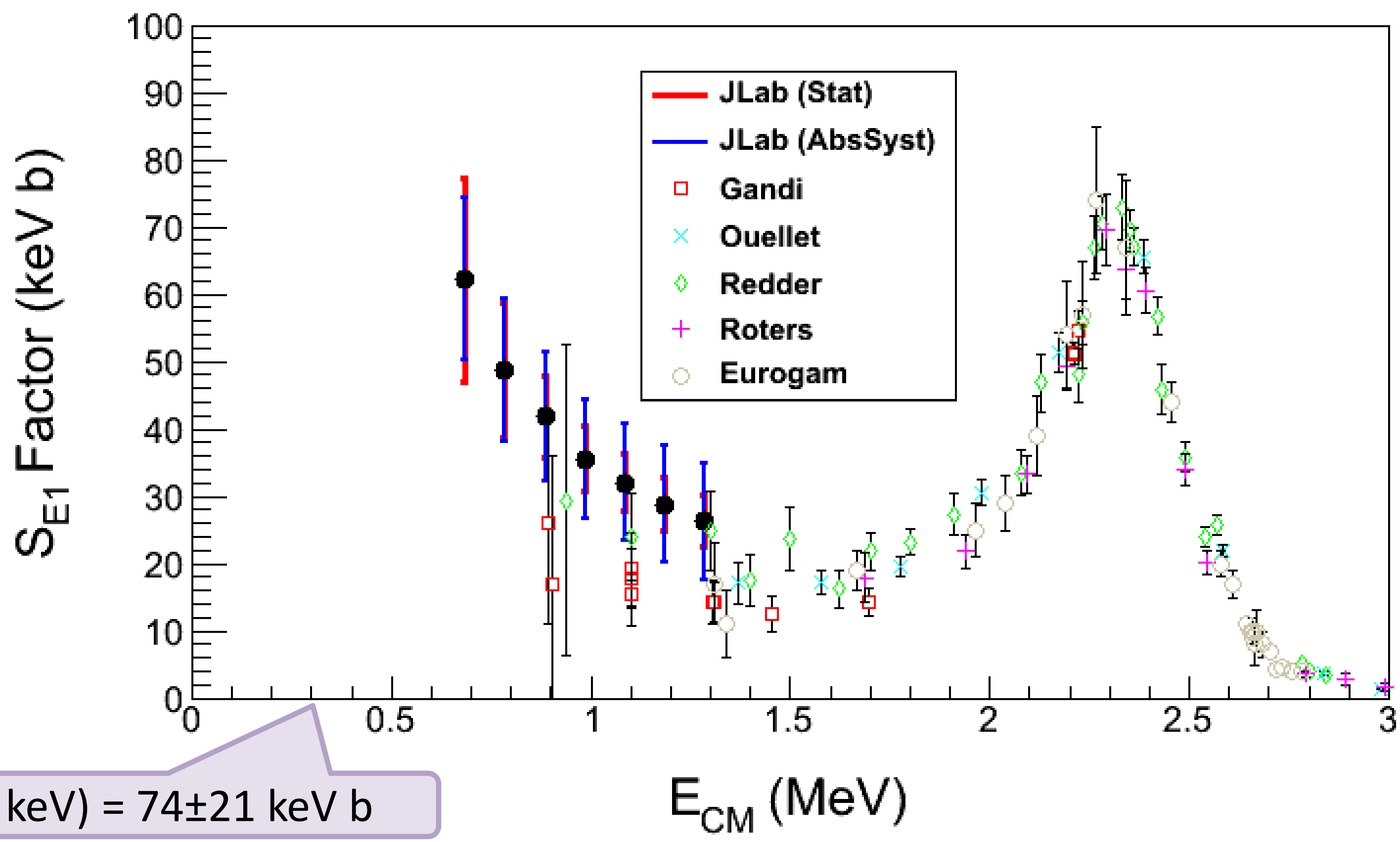
Electron Beam K. E.	Cross Section (nb)	Sys Error (Energy, %)	Sys Error (Total, %)
7.9	0.046	17.4	19.5
8.0	0.185	18.1	21.5
8.1	0.58	18.4	22.9
8.2	1.53	18.9	24.9
8.3	3.49	19.7	27.2
8.4	7.2	20.6	30.0
8.5	13.6	21.6	33.0

Electron Beam K. E.	Gamma Energy (MeV)	E_{CM} (MeV)	Cross Section (nb)	S_{E1} Factor (keV b)	Stat Error (%)	Sys Error (%)
7.9	7.85	0.69	0.046	62.2	24.5	19.5
8.0	7.95	0.79	0.185	48.7	20.7	21.5
8.1	8.05	0.89	0.58	41.8	14.7	22.8
8.2	8.15	0.99	1.53	35.5	13.8	24.9
8.3	8.25	1.09	3.49	32.0	13.3	27.2
8.4	8.35	1.19	7.2	28.8	13.8	30.0
8.5	8.45	1.29	13.6	26.3	14.8	33.0

$^{12}C(\alpha, \gamma)^{16}O$ S-Factor

- Statistical Error: dominated by background subtraction from $^{18}O(\gamma,\alpha)^{14}C$ (depletion = 5,000)

- Systematic Error: dominated by absolute beam energy (δE = 0.1%)



S_{E1}(300 keV) = 74±21 keV b

Relative Systematic Errors

Systematic Error Propagation (1)

- For absolute beam energy uncertainty of δE (= 0.1%) and zero relative beam energy uncertainty:

$$\frac{dy_i}{y_i} = \frac{y_i(E_i + \delta E) - y_i(E_i)}{y_i(E_i)}$$

$$\frac{dN_{ij}}{N_{ij}} = \frac{N_{ij}(E_i + \delta E) - N_{ij}(E_i)}{N_{ij}(E_i)}$$

$$E_0 = 7.8 + \delta E$$

$$E_i = E_0 + i\Delta$$

E_i (MeV)	dy_i/y_i (%)	$d\sigma_i/\sigma_i$ (%)
7.9	12.5	12.6
8.0	10.8	10.5
8.1	9.3	9.1
8.2	8.0	7.1
8.3	7.0	6.3
8.4	6.3	5.8
8.5	5.6	5.2

This is the cross section dependence on energy

- Accounted for dN_{ij} due to energy error when calculating dy_i

$$\approx \frac{\delta E}{i\Delta}$$

$$\left[dN_{ij} / N_{ij}\right] = \begin{bmatrix} 0.100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.058 & 0.050 & 0 & 0 & 0 & 0 & 0 \\ 0.041 & 0.039 & 0.033 & 0 & 0 & 0 & 0 \\ 0.031 & 0.031 & 0.029 & 0.025 & 0 & 0 & 0 \\ 0.025 & 0.025 & 0.025 & 0.023 & 0.020 & 0 & 0 \\ 0.021 & 0.021 & 0.021 & 0.021 & 0.020 & 0.017 & 0 \\ 0.018 & 0.018 & 0.018 & 0.018 & 0.018 & 0.017 & 0.022 \end{bmatrix}$$

- With:

$$[B] = [N]^{-1}$$

$$[\sigma] = [B] \bullet [Y]$$

- Then:

$$\left[d\sigma^2\right] = [B] \bullet \left(\left[dY^2\right] + \left[dN^2\right] \bullet \left[\sigma^2\right]\right) \bullet [B]^T$$

- Where:

Note: Correlation Coefficient =1

$$\mathrm{var}(y_i, y_i) = (dy_i)^2$$
$$\mathrm{cov}(y_i, y_j) = \rho_{ij} dy_i dy_j$$

$$\left[dY^2\right] = \begin{bmatrix} (dy_1)^2 & dy_1 dy_2 & \cdots & dy_1 dy_n \\ dy_2 dy_1 & (dy_2)^2 & \cdots & dy_n dy_n \\ \vdots & \ddots & \ddots & \vdots \\ dy_n dy_1 & dy_n dy_2 & \cdots & (dy_n)^2 \end{bmatrix}$$

$$\left[d\sigma^2\right] = \begin{bmatrix} d\sigma_1^2 & \mathrm{cov}(\sigma_1, \sigma_2) & \cdots & \mathrm{cov}(\sigma_1, \sigma_n) \\ \mathrm{cov}(\sigma_2, \sigma_1) & d\sigma_2^2 & \cdots & \mathrm{cov}(\sigma_2, \sigma_n) \\ \vdots & \ddots & \ddots & \vdots \\ \mathrm{cov}(\sigma_n, \sigma_1) & \mathrm{cov}(\sigma_n, \sigma_2) & \cdots & d\sigma_n^2 \end{bmatrix}$$

$$
\left[dN^{2}\right]=\begin{bmatrix}
\left(dN_{11}\right)^{2} & 0 & \cdots & 0 \\
\left(dN_{21}\right)^{2} & \left(dN_{22}\right)^{2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\left(dN_{n1}\right)^{2} & \left(dN_{n2}\right)^{2} & \cdots & \left(dN_{nn}\right)^{2}
\end{bmatrix}
$$

$$
\left[\sigma^{2}\right]=\begin{bmatrix}
\sigma_{1}^{2} & 0 & \cdots & 0 \\
0 & \sigma_{2}^{2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_{n}^{2}
\end{bmatrix}
$$

Systematic Error Propagation (2)

$$\left(d\sigma_i\right)^2 \cong \frac{1}{N_{ii}^2}\Bigg[dy_i^2 - 2dy_i\sum_{j=1}^{i-1} N_{ij}d\sigma_j + \sum_{j=1}^{i-1}\left(N_{ij}d\sigma_j\right)^2 + \sum_{k=1}^{i-1}\sum_{l=1}^{i-1} N_{ik}\,\mathrm{cov}(\sigma_k,\sigma_l)N_{il} + \sum_{j=1}^{i-1}\left(dN_{ij}\sigma_j\right)^2 + \left(dN_{ii}\sigma_i\right)^2\Bigg]$$

$$\mathrm{cov}(y_i, y_j) \neq 0,$$
$$\mathrm{cov}(\sigma_i, \sigma_j) \neq 0$$

Other Relative Systematic Errors

Beam Current, $\delta I/I$	3%
Photon Flux, $\delta \varphi/\varphi$	5%
Radiator Thickness, $\delta R/R$	3%
Bubble Chamber Thickness, $\delta T/T$	3%
Bubble Chamber Efficiency, ε	5%

Simulation

- Then:

$$(dy_i)^2 = (dy_i(\delta E))^2 + \left[\left(\frac{\delta I}{I}\right)^2 + \left(\frac{\delta R}{R}\right) + \left(\frac{\delta T}{T}\right)^2 + \varepsilon^2\right] y_i^2$$

$$(dN_{ij})^2 = \left(\frac{\delta \phi}{\phi}\right)^2 N_{ij}^2$$

Relative Systematic Errors Results

Electron Beam K. E.	Cross Section (nb)	Sys Error (Energy, %)	Sys Error (Total, %)
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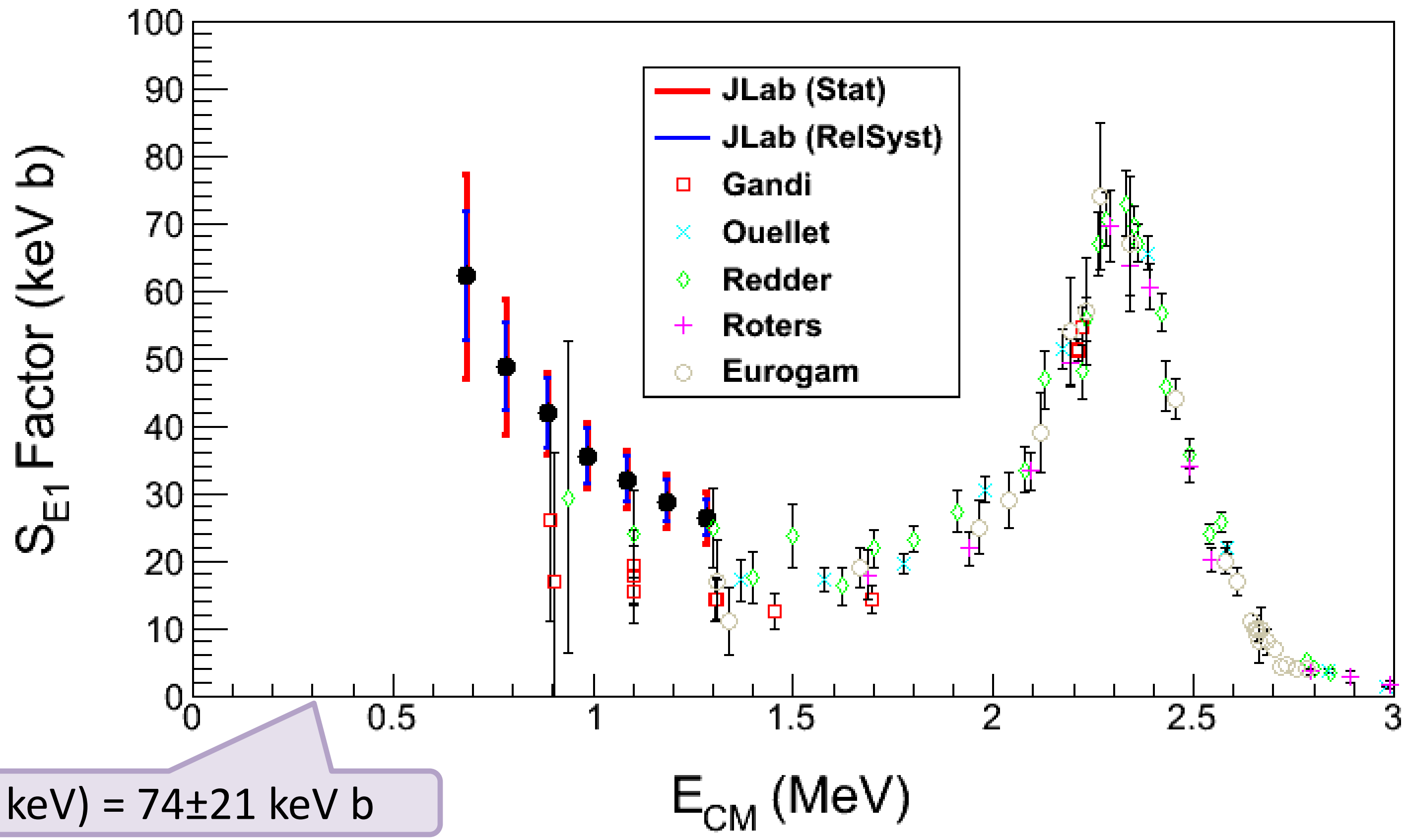
8.3	3.49	6.0	10.7
8.4	7.2	5.3	10.5
8.5	13.6	4.7	10.1

Note: Relative systematic errors do not get amplified in PL Unfolding

$^{12}C(\alpha, \gamma)^{16}O$ S-Factor

- Statistical Error: dominated by background subtraction from $^{18}O(\gamma,\alpha)^{14}C$ (depletion = 5,000)

- Relative Systematic Errors



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