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# A MODIFICATION TO THE PENFOLD–LEISS METHOD OF CROSS-SECTION UNFOLDING

D.J.S. FINDLAY
*Nuclear Physics Division, A.E.R.E., Harwell, Oxon., England*

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The standard Penfold–Leiss method of unfolding a photonuclear cross-section from a measured bremsstrahlung yield curve is shown to fail if the cross-section is unfolded in photon energy bins whose width is comparable with the range of any smearing of the bremsstrahlung spectrum near the endpoint. In the case described in the text the smearing is due to the energy spread of electrons striking the bremsstrahlung radiator. A simple modification to the Penfold–Leiss prescription is described which overcomes this failure.

## 1. Introduction

Photonuclear physics experiments continue to be performed using bremsstrahlung as the source of photons. Bremsstrahlung photons are especially useful for photofission measurements near threshold where the energies of interest are such that the well known technique of positron annihilation in flight producing quasi-monochromatic photons suffers from low intensity and poor resolution. However the continuous range of photon energies produced by bremsstrahlung means that the cross-section (normally the quantity of interest) is not measured directly – instead the cross-section must be unfolded from the measured yields. There are available several different prescriptions for unfolding cross-sections from bremsstrahlung yield curves (see, for example, refs. 1–3) and much work has been done on the properties of the unfolded cross-section (see, for example, refs. 4–7). However, the unfolding procedures have often been considered in isolation from the photon energy spectrum of the bremsstrahlung beam used experimentally. It is the purpose of this paper to point out that for photofission cross-sections measured near threshold with bremsstrahlung from an electron linear accelerator the details of the shape of the bremsstrahlung spectrum must be considered explicitly in the unfolding procedure. The unfolding prescription considered in this paper is the Penfold–Leiss prescription [1]; this is the most direct of the several available prescriptions.

## 2. The Penfold–Leiss prescription

The relation between the cross-section $\sigma$ as a function of photon energy $k$ and the measured yield $Y$ at any bremsstrahlung endpoint energy $E$ is the integral equation

$$Y(E)=\int_{k_{\mathrm{thr}}}^{E} N(E,k)\sigma(k)\,\mathrm{d}k, \tag{1}$$

where $N(E, k)$ is the bremsstrahlung spectrum for electron or endpoint energy $E$ as a function of photon energy $k$ and $k_{\mathrm{thr}}$ is the threshold energy of the cross-section. [In eq. (1), the response of the bremsstrahlung dose measuring device has been absorbed into $Y(E)$, and the attenuation of the bremsstrahlung beam in the photonuclear target and any filters has been absorbed into $\sigma(k)$.] The yield $Y(E)$ is not measured as a continuous function of $E$ but at a series of energies $E=E_1, E_2,\ldots, E_n$ which will be assumed to be equally spaced with spacing $\Delta$ (i.e. $E_i-E_{i-1}=\Delta$, $i=2, n$). The Penfold–Leiss [1] prescription is to take appropriate linear combinations

$$\sigma_i=\sum_{j=1}^{i} B_{ij}Y(E_j) \tag{2}$$

of the yields $Y(E_j)$ such that

$$\sigma_i \equiv \sigma[E_i-O(\Delta)], \tag{3}$$

where $O(\Delta)$ is a quantity of the order of $\Delta$. The coefficients $B_{ij}$ are chosen from

$$\Delta\sum_{j=1}^{i} B_{ij}N\left(E_j, E_{i'}-\frac{\Delta}{2}\right)=\delta_{ii'} \tag{4}$$

[the coefficients $B_{ij}$ form a matrix which is the inverse of the matrix $N_{ij}=\Delta N(E_i, E_j-\frac{1}{2}\Delta)$]. This produces unfolded cross-section points $\sigma_1, \sigma_2,\ldots, \sigma_n$ spaced apart in energy by a bin width $\Delta$. Writing $Y(E_j)$ explicitly as in eq. (1) and recognising that $N(E_j, k)=0$ for $k>E_j$,

$\sigma_i$ becomes

$$\sigma_i = \int_{k_{thr}}^{E_i} \left[ \sum_{j=1}^{i} B_{ij} N(E_j, k) \right] \sigma(k)\, dk$$
$$= \int_{k_{thr}}^{E_i} T_i(k) \sigma(k)\, dk, \qquad (5)$$

where

$$T_i(k) \equiv \sum_{j=1}^{i} B_{ij} N(E_j, k) \qquad (6)$$

is the weighting function. The weighting function, composed of a linear combination of bremsstrahlung spectra, should satisfy the following criteria: (1) it should be identically zero for $k > E_i$, (2) it should have a peak of width $\sim \Delta$ at $k = E_i - O(\Delta)$, (3) it should be significantly different from zero only for $E_i - n\Delta \le k \le E_i$ where $n \sim 3$ and (3) it should have unit area. Usually the unfolded cross-section point $\sigma_i$ is assigned an energy $\bar{k}_i = \int k T_i(k)\, dk$, where $\bar{k}$ is the centroid energy of the weighting function $T_i(k)$.

If $N(E, k)$ is taken to be the Schiff integrated-over-angle bremsstrahlung spectrum [8] and if the yield point spacing and bin width is $\Delta = 100$ keV (a resolution of 100 keV is desirable in a measured photofission cross-section), the weighting function $T_i^S(k)$ is typically as shown in fig. 1. [In fig. 1 $E_i = 6.5$ MeV and $E_1 = 5.5$ MeV, but apart from the shift in energy with $i$ the shape of $T_i^S(k)$ as a whole is very nearly independent of $i$ except for $i \le 3$.] Since most strength is centred around $k \cong E_i - \frac{1}{2}\Delta$, and there is very little strength elsewhere, it is indeed apparent from eq. (5) that $\sigma_i \cong \sigma(E_i - \frac{1}{2}\Delta)$ as prescribed by eq. (3). (In ref. 1 Penfold and Leiss show a similar weighting function for the much larger bin width $\Delta = 1$ MeV.)

For photofission measurements near threshold using an electron linear accelerator, the calculation of the spectrum of tightly collimated bremsstrahlung at 0° striking the photonuclear target is described in ref. 9. The calculation is based on recent partial wave calculations of the integrated-over-angle bremsstrahlung cross-section, and includes energy losses, straggling and multiple scattering of the electrons in the bremsstrahlung radiator, the variation of the bremsstrahlung cross-section angular distribution with photon energy and an incident electron energy spectrum appropriate for the analysed beam from an electron linear accelerator. If bremsstrahlung spectra are computed according to ref. 9 for the Harwell electron linear accelerator where the electron energy spectrum is analysed to $\sim \pm 1\%$, and if again the bin width is 100 keV, then choosing the coefficients of the linear combination [eq. (2)] from eq. (4) results in the typical weighting function $T_i^H(k)$ shown in fig. 2. It is obvious that $T_i^H(k)$ is quite unsatisfactory as a weighting function – the wildly unstable oscillatory behaviour means that the four

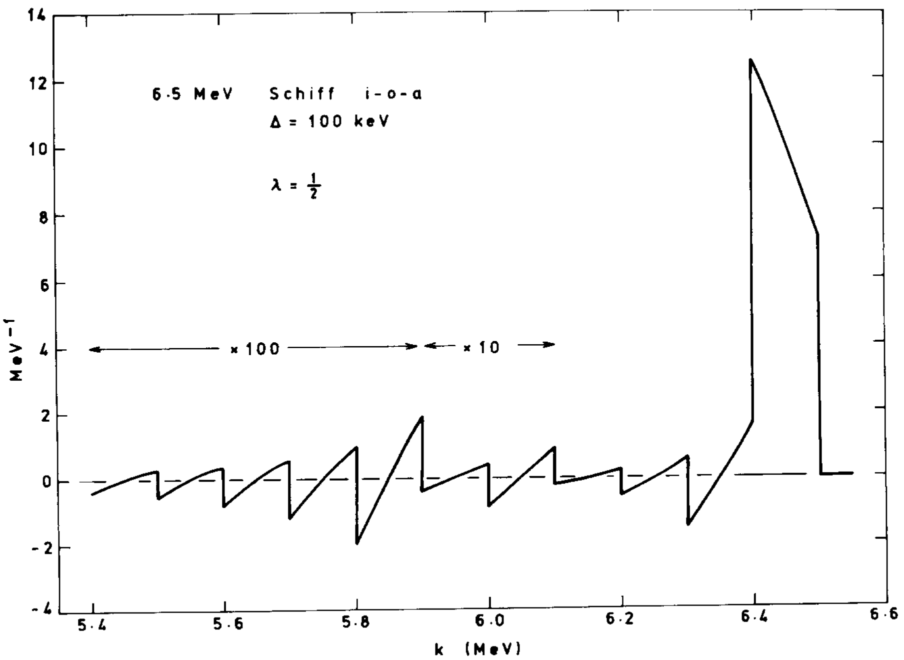


Fig. 1. The 6.5 MeV weighting function computed according to the Penfold–Leiss prescription for the Schiff integrated-over-angle bremsstrahlung spectrum for gold ($Z = 79$) using 100 keV bins. (The prescription of the weighting function by eq. (4) is denoted by $\lambda = \frac{1}{2}$.)

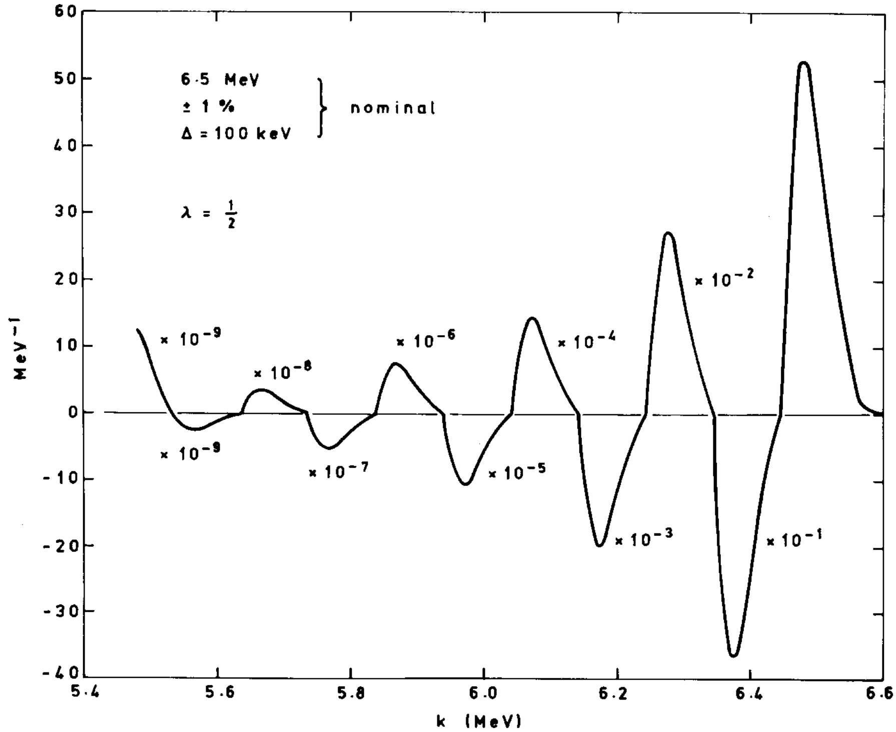


Fig. 2. The 6.5 MeV weighting function computed according to the Penfold–Leiss prescription for bremsstrahlung from a gold ($Z = 79$) radiator computed for the Harwell electron linear accelerator with a nominal $\pm 1\%$ energy spread using 100 keV bins. Note the multiplicative factors.

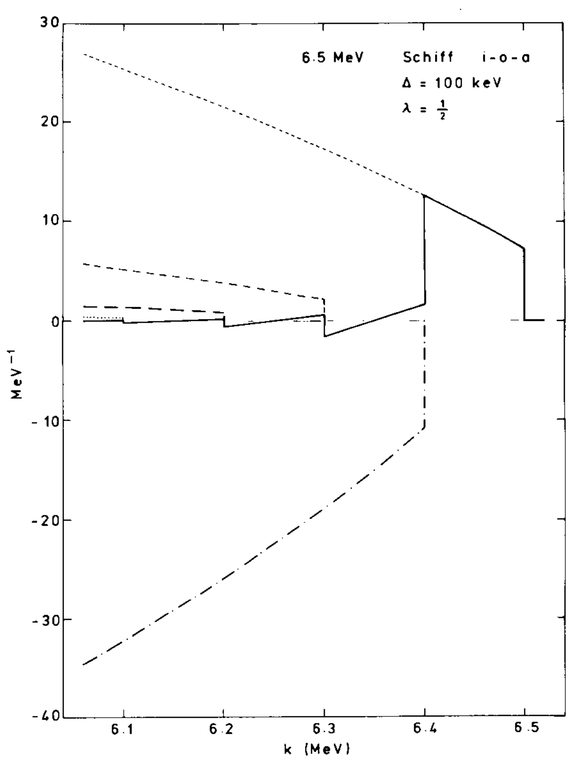


Fig. 3. The construction of the weighting function shown in fig. 1. *Dashed lines*, component bremsstrahlung spectra; *solid line*, resultant weighting function.

criteria mentioned above are not met – so the linear combination eq. (2) does not represent the cross-section $\sigma(k)$ at $k \equiv E_i$ as prescribed by eq. (3); in fact, the unfolded cross-section points $\sigma_1, \sigma_2, \ldots, \sigma_n$ [eq. (2)] oscillate wildly. [For the bremsstrahlung spectra leading to $T_i^{\mathrm{H}}(k)$ which result from electrons with a finite energy spread striking the bremsstrahlung radiator than monochromatic electrons, $\bar{E}_i$ is used to denote the mean electron energy; the highest photon energy $k$ in fig. 2 is therefore $E_i' = (1 + \frac{1}{2}\epsilon)\ \bar{E}_i$, where $\epsilon$ is the fractional maximum energy spread of the electron beam, and in addition the actual bin width becomes $\Delta' = (1 + \frac{1}{2}\epsilon)$ $\Delta$, $\Delta'$ being the separation of neighbouring bremsstrahlung spectra endpoint energies and $\Delta$ being the separation of the mean energies $\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_n$. It is shown in ref. 9 that $\epsilon = 0.030$ for a nominal $\pm 1\%$ energy spread.]

The wildly unstable oscillatory behaviour of the weighting function is due to the detailed shape of the bremsstrahlung spectrum near the endpoint and is understood by considering the construction of the weighting function specified by eq. (4). For the Schiff integrated-over-angle bremsstrahlung spectrum the components of the weighting function $T_i^S(k)$ prescribed by the coefficients $B_{ij}$ from eq. (4) are shown in fig. 3. Eq. (4) forces $T_i(k)$ to have a value $1/\Delta$ at $k = E_i - \frac{1}{2}\Delta$ and to pass through zero at $k = E_{i-1} - \frac{1}{2}\Delta$, $E_{i-2} - \frac{1}{2}\Delta$, .... The weighting function $T_i(k)$ in the uppermost bin $\Delta_i$ between $E_i$ and $E_{i-1}$ is just the highest endpoint energy bremsstrahlung spectrum $N(E_i, k)$ re-normalised to have a value $1/\Delta$ at the centre $k = E_i - \frac{1}{2}\Delta$ of the uppermost bin $\Delta_i$. The shape of $T_i(k)$ in the second uppermost bin $\Delta_{i-1}$ between $E_{i-1}$ and $E_{i-2}$ is the shape of the difference $N(E_i, k) - \alpha_1 N(E_{i-1}, k)$ between the two highest endpoint energy bremsstrahlung spectra $N(E_i, k)$ and $N(E_{i-1}, k)$ with the second highest endpoint energy bremsstrahlung spectrum $N(E_{i-1}, k)$ re-normalised (by the multiplicative factor $\alpha_1$) so that the difference is zero at the centre $k = E_{i-1} - \frac{1}{2}\Delta$ of the second uppermost bin $\Delta_{i-1}$. The shape of $T_i(k)$ in the bin $\Delta_{i-l}$ is derived similarly to the shape in the bin $\Delta_{i-1}$ but with the sum $N(E_i, k) - \alpha_1 N(E_{i-1}, k) + \alpha_2(E_{i-2}, k) + \ldots + \alpha_l N(E_{i-l}, k)$ going to zero at the centre of the bin $\Delta_{i-l}$. From fig. 3 it can be seen that the process converges quite rapidly; apart from the second highest energy bremsstrahlung spectrum the strengths $\alpha_l$ of the bremsstrahlung spectra $\alpha_l N(E_{i-l}, k)$ decrease quite rapidly as $l$ increases.

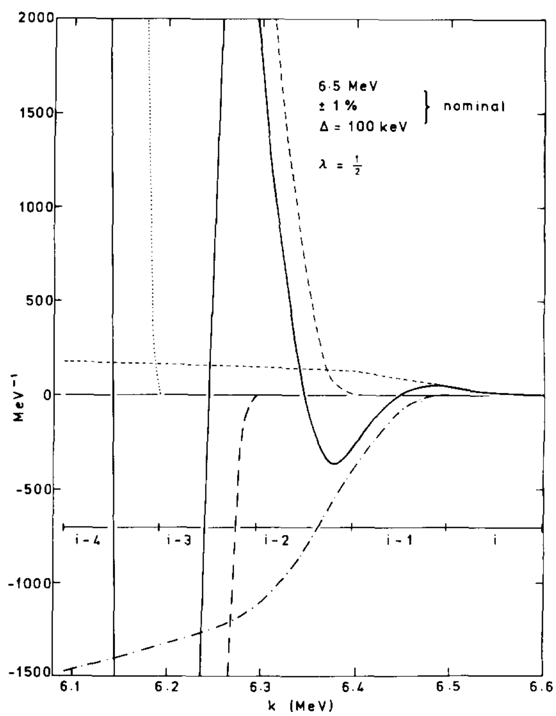


Fig. 4. The construction of the weighting function shown in fig. 2. *Dashed lines,* component bremsstrahlung spectra; *solid line,* resultant weighting function. The bins $\Delta'$ (see text) are labelled $i$, $i$-1, $i$-2, ... .

However for bremsstrahlung spectra computed according to ref. 9 for the Harwell electron linear accelerator with a nominal $\pm 1\%$ energy spread, the components of the weighting function $T_i^H(k)$ prescribed by the coefficients $B_{ij}$ from eq. (4) are shown in fig. 4. The increasing strength $\alpha_l$ of the bremsstrahlung spectra $\alpha_l N(\bar{E}_{i-l}, k)$ with $l$ – i.e. the wildly unstable oscillatory behaviour – can be seen to be due to the change in slope of the bremsstrahlung spectrum within an energy $\sim \Delta$ of the endpoint. To make the weighting function $T_i^H(k)$ go to zero at the centre of the bin $\Delta'_{i-1}$ in fig. 4 the bremsstrahlung spectrum $\alpha_1 N(\bar{E}_{i-1}, k)$ must be large and negative; however this means that to make $T_i^H(k)$ go to zero at the centre of the bin $\Delta'_{i-2}$, $\alpha_2 N(\bar{E}_{i-2}, k)$ must be even larger and positive; but now to make $T_i^H(k)$ go to zero at the centre of the bin $\Delta'_{i-3}$, $\alpha_3 N(\bar{E}_{i-3}, k)$ has to be yet larger and negative; extension of this procedure to bins $\Delta'_{i-4}$, $\Delta'_{i-5}$, ... demonstrates the build-up of the wildly unstable oscillatory behaviour. The offending change in the slope of the bremsstrahlung spectrum near the endpoint is mostly due to the nominal $\pm 1\%$ energy spread of the electron beam incident on the radiator.

The shape of the bremsstrahlung spectrum computed according to ref. 9 for the Harwell electron linear accelerator with a nominal $\pm 1\%$ energy spread is at most $\sim 10\%$ different from the shape of the Schiff integrated-over-angle bremsstrahlung spectrum except near the endpoint, but as indicated above the cross-sections unfolded from the same yield curve using these two bremsstrahlung spectra are very different. This is contrary to the suggestion of Penfold and Leiss [1] that the difference between the two cross-sections unfolded from the same yield curve using these two bremsstrahlung spectra should amount to a small energy shift and re-normalisation. The present large difference is due to the fact that the present bin width $\Delta$ is comparable to the energy region $\Delta k_{\text{endpoint}}$ near the endpoint over which the bremsstrahlung spectra differ i.e. $\Delta \sim \Delta k_{\text{endpoint}}$, whereas Penfold and Leiss envisaged $\Delta \gg \Delta k_{\text{endpoint}}$.

## 3. Modification of the Penfold–Leiss prescription

The problem demonstrated by fig. 4 can be resolved simply by choosing another set of coefficients $B_{ij}$ for the linear combination eq. (2). Any set of coefficients which produces a satisfactory weighting function – meeting the four criteria mentioned above – is acceptable. The simplest alteration to the Penfold–Leiss prescription of eq. (4) for the coefficients $B_{ij}$ is to replace the $\Delta/2$ in eq. (4) by $\lambda\Delta$

$$\Delta \sum_{j=1}^{i} B_{ij} N(E_j, E_{i'} - \lambda\Delta) = \delta_{ii'}, \tag{7}$$

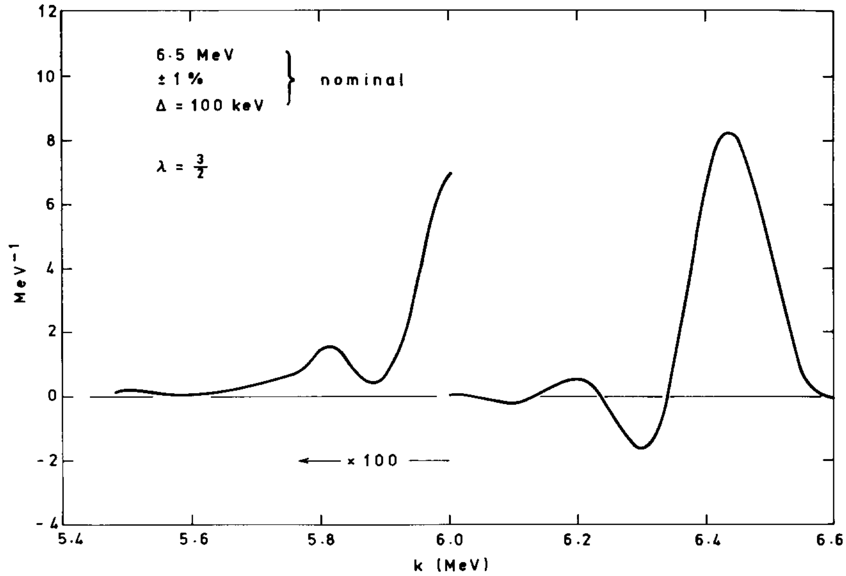


Fig. 5. As fig. 2 but for $\lambda = 3/2$.

and to treat $\lambda$ as a parameter. For $\Delta = 100$ keV the best choice was found to be $\lambda = 3/2$, and figs. 5 and 6 show the resultant weighting function $T_i^H(k;\ \Delta = 100$ keV, $\lambda = 3/2)$ and its components respectively ($\bar{E}_i = 6.5$ MeV, $\bar{E}_1 = 5.5$ MeV). The fwhm of $T_i^H(k;$ 100 keV, 3/2) is ~

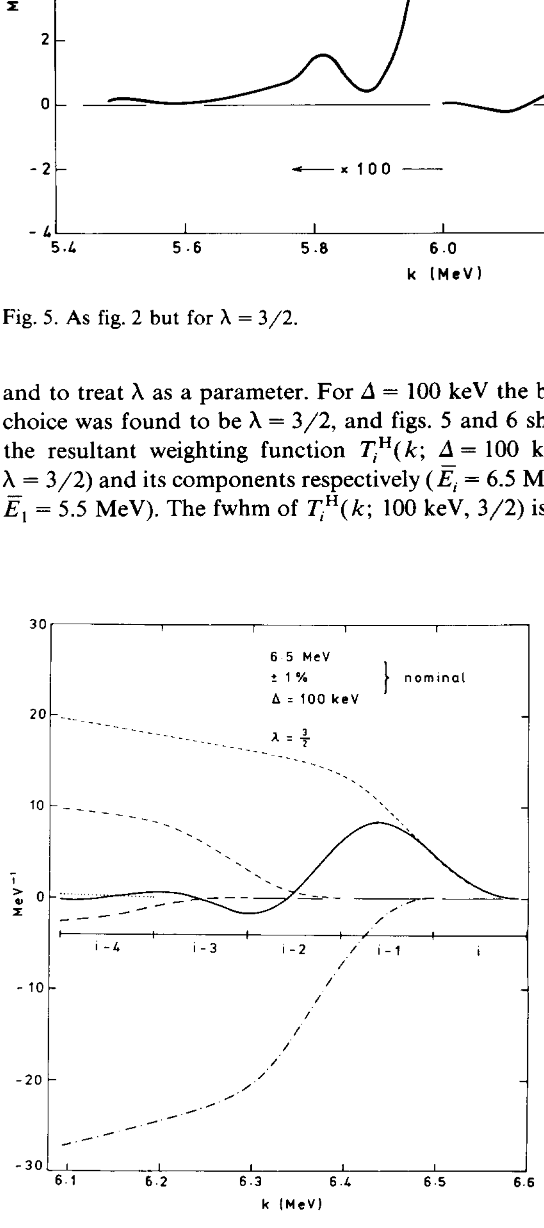


Fig. 6. Construction of the $\lambda = 3/2$ weighting function shown in fig. 5. *Dashed lines*, component bremsstrahlung spectra; *solid line*, resultant weighting function. The bins $\Delta'$ are labelled as in fig. 4.

130 keV which is the same as the energy spread of the electron beam incident on the radiator. The fwhm is relatively insensitive to small changes in $\lambda$ about $\lambda = 3/2$. Weighting functions were also computed for $\Delta = 50$ keV for bremsstrahlung spectra computed according to ref. 9 for the Harwell electron linear accelerator with a nominal $\pm 1\%$ energy spread. The best value of $\lambda$ was found to be $\lambda \cong 5/2$. Best values of $\lambda$ are given roughly by $\lambda\Delta = \Delta E_{\text{spread}}$ where $\Delta E_{\text{spread}}$ is the energy spread of the electron beam incident on the bremsstrahlung radiator.

The normalisation integrals of the weighting functions $\int T_i^H(k;$ 100 keV, 3/2) d$k$ are typically within 1%of unity, and the centroid energies $\bar{k}_i = \int k T_i^H(k;$ 100 keV, 3/2) d$k$ are typically within 10 keV of $E_i' - \lambda\Delta'$. Although as mentioned above the unfolded cross-section point $\sigma_i$ is usually assigned an energy $\bar{k}_i$ so that $\sigma_i = \sigma(\bar{k}_i)$, this is of course strictly true only for a uniform cross-section. *

## 4. Conclusions

The standard Penfold–Leiss method of unfolding a cross-section from a yield curve has been shown to generate spurious results when using bremsstrahlung produced by electrons whose energy spread is comparable to the bin width $\Delta$ used in the unfolding procedure.

* It may be noted in passing that the centroid energy $\bar{k}_i$ of the Penfold–Leiss $\lambda = \frac{1}{2}$ weighting function for the Schiff integrated-over-angle bremsstrahlung spectrum is reasonably close to, but not equal to, the value $E_i - \frac{1}{2}\Delta$ usually assumed.

Similarly spurious results will also occur if the bin width is comparable to smearing near the bremsstrahlung spectrum endpoint due to electron energy losses in the bremsstrahlung radiator. A relatively simple change to the coefficients defining the Penfold–Leiss weighting function has been made which prevents the generation of these spurious results.

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