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Lorentz Transform of an Arbitrary Force Field on a Particle in its Rest Frame using the Hamilton-Lagrangian Formalism

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Abstract:

A formalism is described for deriving the Lagrangian, Hamiltonian, and canonical momentum of a particle interacting with an arbitrary force field, based on the interaction energy in the rest frame of the particle. Transformation to the laboratory frame is described and the resulting force on the particle derived.

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1 Rest Frame

1.1 Lagrangian

In the rest frame of a particle of mass m, its Lagrangian is the difference between its kinetic energy and the sum of its restmass energy and its energy of interaction with the external field [1)]

$$L = E_{kin} - (mc^2 + E_{field}) \quad . \tag{1}$$

In the particle's true rest frame (TRF; double-starred quantities), its kinetic energy is zero so that

$$L^{**} = -mc^2 - \varphi^{**} \tag{2}$$

where φ^{**} is the interaction energy of the particle with the external field in the TRF. In order to describe the velocity dependence of the Lagrangian for very small velocities, we define a "fixed rest frame" (FRF; starred quantities) which moves with a fixed velocity $c\vec{\beta_0}$ with respect to the laboratory frame (LF). The TRF moves at an infinitesimal velocity $c\vec{\beta^*}$ with respect to the FRF and with velocity $c\vec{\beta}$ with respect to the LF.

The FRF Lagrangian L^* is related to L^{**} by

$$L^* = L^{**}/\gamma^* \tag{3}$$

where $\gamma^* \equiv 1/\sqrt{1 - \vec{\beta^*}^2}$, since $\gamma^* L^*$ is a Lorentz invariant [2)]. Thus, for a particle of mass m, we have

$$L^* = -mc^2/\gamma^* - \varphi^{**}/\gamma^* \tag{4}$$

where the first part, $L^*_\mu = -mc^2/\gamma^*$, represents the particle's motion and mass energy, and the second part, $L^*_\varphi = -\varphi^*(t^*, \vec{x^*}, \vec{\beta^*})$, the particle-field interaction, and where we have defined $\varphi^*(t^*, \vec{x^*}, \vec{\beta^*}) \equiv \varphi^{**}/\gamma^*$.

1.2 Canonical Momentum

The canonical momentum $\vec{P^*} = \vec{P^*_\mu} + \vec{P^*_\varphi}$ conjugate to the coordinates $\vec{x^*}$ associated with a Lagrangian L^* is defined [1)] as the partial derivative of the Lagrangian with respect to the particle's velocity $c\vec{\beta^*}$. Its parts

$$cP^*_{\mu,k} = \partial L^*_\mu / \partial \beta^*_k = \gamma^* \beta^*_k mc^2 \tag{5}$$

and

$$cP^*_{\varphi,k} = \partial L^*_\varphi / \partial \beta^*_k = -\partial \varphi^* / \partial \beta^*_k \quad , \tag{6}$$

without being canonical momenta in themselves, represent the particle motion and the external force field, respectively.

The Taylor expansion of φ^* in $\vec{\beta^*}$ about $\vec{\beta^*} = 0$ is

$$\varphi^*(t^*, \vec{x^*}, \vec{\beta^*}) = \varphi^*_0(t^*, \vec{x^*}) - c\beta^*_m P^*_{\varphi 0,m} - \frac{1}{2}\beta^*_m \beta^*_\ell Q^*_{\ell m} \quad + \mathcal{O}(\beta^{*3}) \tag{7}$$

where

$$\begin{aligned} \varphi^*_0 &\equiv \varphi^*(t^*, \vec{x^*}, \vec{\beta^*} = 0) = \varphi^{**}(\vec{\beta^*} = 0) && (8) \\ cP^*_{\varphi 0,m} &\equiv -\left[\frac{\partial \varphi^*}{\partial \beta^*_m}\right]_{\vec{\beta^*}=0} ; \quad Q^*_{\ell m} \equiv -\left[\frac{\partial^2 \varphi^*}{\partial \beta^*_m \partial \beta^*_\ell}\right]_{\vec{\beta^*}=0} && (9) \end{aligned}$$

and we follow the convention of summing over double indices.

Thus we obtain the following expressions for the field part of the canonical momentum and its derivative with respect to $\beta*$ in the FRF:

$$\begin{aligned} cP^*_{\varphi,k}(t^*, \vec{x^*}, \vec{\beta^*}) &= cP^*_{\varphi 0,k}(t^*, \vec{x^*}) + \beta^*_\ell Q^*_{\ell k}(t^*, \vec{x^*}) \quad + \mathcal{O}(\vec{\beta^*}^2); && (10) \\ c\partial P^*_{\varphi,k} / \partial \beta^*_n &= Q^*_{nk} \quad . && (11) \end{aligned}$$

1.3 Hamiltonian

The Hamiltonian H^* is defined as the sum of the kinetic and interaction energy [1)]. It is related to the Lagrangian and the canonical momentum by $H^* = c\vec{\beta^*}\vec{P^*} - L^*$ and separates like the Lagrangian into the parts H^*_μ associated with particle motion and H^*_φ with the external field:

$$\begin{aligned} H^*_\mu &= c\vec{\beta^*}\vec{P^*_\mu} - L^*_\mu = mc^2\gamma^*\beta^*_m\beta^*_m + mc^2/\gamma^* = \gamma^* mc^2 \quad ; && (12) \\ H^*_\varphi &= c\vec{\beta^*}\vec{P^*_\varphi} - L^*_\varphi = \varphi^*_0(t^*, \vec{x^*}) + \frac{1}{2}\beta^*_m\beta^*_\ell Q^*_{\ell m}(t^*, \vec{x^*}) \quad . && (13) \end{aligned}$$

The Hamiltonian, expressed as a function of $t^*, \vec{x^*}, \vec{\beta^*}$ and of $t^*, \vec{x^*}, \vec{P^*}$ is therefore

$$\begin{aligned} H^*(t^*, \vec{x^*}, \vec{\beta^*}) &= H^*_\mu + H^*_\varphi = \gamma^* mc^2 + \varphi^*_0(t^*, \vec{x^*}) + \frac{1}{2}\beta^*_m\beta^*_\ell Q^*_{\ell k}(t^*, \vec{x^*}) \quad ; && (14) \\ H^*(t^*, \vec{x^*}, \vec{P^*}) &= \sqrt{m^2c^4 + c^2[\vec{P^*} - \vec{P^*_\varphi}(t^*, \vec{x^*})]^2} + \varphi^*_0(t^*, \vec{x^*}) + \frac{1}{2}\beta^*_m\beta^*_\ell Q^*_{\ell k}(t^*, \vec{x^*}) \, . && (15) \end{aligned}$$

2 Lorentz Boost to the Laboratory Frame

Since the product $\gamma^* L^*$ is invariant under Lorentz transformations [2], the Lagrangian L, the canonical momentum $\vec{P}$, and the Hamiltonian H in the LF are [1]

$$L = (\gamma^*/\gamma)L^*; \quad cP_k = c\partial L/\partial\beta_k; \quad H = c\vec{P}\vec{\beta} - L \ . \tag{16}$$

The particle motion parts of the LF Lagrangian L_μ, of the canonical LF momentum $cP_{\mu,k} \equiv \partial L_\mu/\partial\beta_k$, and of the LF Hamiltonian $H_\mu = c\vec{P}_\mu\vec{\beta} - L_\mu$ are

$$L_\mu = -mc^2/\gamma; \quad c\vec{P}_\mu = mc^2\gamma\vec{\beta}; \quad H_\mu = mc^2\gamma \ . \tag{17}$$

The field interaction part of the LF Lagrangian L_φ is

$$L_\varphi = -\frac{\gamma^*}{\gamma}\varphi^* = -\frac{\gamma^*}{\gamma}\varphi_0^* + \frac{\gamma^*}{\gamma}cP^*_{\varphi0,m}\beta^*_m + \frac{1}{2}\frac{\gamma^*}{\gamma}\beta^*_m\beta^*_\ell Q^*_{\ell m} \quad + \mathcal{O}(\beta^{*3}) \tag{18}$$

which, according to eqns. (120) and (121), becomes

$$L_\varphi = \gamma_0(\vec{\beta}\vec{\beta}_0 - 1)\varphi_0^* + c\vec{P}^*_{\varphi0}\vec{\beta} + \gamma\phi(\vec{\beta}_0\vec{\beta})(c\vec{P}^*_{\varphi0}\vec{\beta}_0) - \gamma_0(c\vec{P}^*_{\varphi0}\vec{\beta}_0) + \frac{1}{2\gamma_0}\beta^*_m\beta^*_\ell Q^*_{\ell m} \quad + \mathcal{O}(\beta^{*3}) \tag{19}$$

where $\phi \equiv \gamma_0/(\gamma_0+1)$. The corresponding part of the LF momentum $P_{\varphi,k} = \partial L_\varphi/\partial\beta_k$ is then, using eqn. (124),

$$cP_{\varphi,k} = \gamma_0\beta_{0,k}\varphi_0^* + cP^*_{\varphi0,k} + \gamma_0\phi\beta_{0,k}(c\vec{P}^*_{\varphi0}\vec{\beta}_0) + [\delta_{mk} + \gamma_0\phi\beta_{0,k}\beta_{0,m}]\,\beta^*_\ell Q^*_{\ell m} + \frac{\gamma_0}{2}\beta_{0,k}\beta^*_m\beta^*_\ell Q^*_{\ell m} \quad + \mathcal{O}(\beta^{*3}) \tag{20}$$

and the field part of the LF Hamilronian H_φ is

$$H_\varphi = \gamma_0\varphi_0^* + \gamma_0(c\vec{P}^*_{\varphi0}\vec{\beta}_0) + \gamma_0\beta_{0,m}\beta^*_\ell Q^*_{\ell m} + \frac{\gamma_0}{2}\beta^*_m\beta^*_\ell Q^*_{\ell m} \quad + \mathcal{O}(\beta^{*3}) \ . \tag{21}$$

If we rewrite eqns. (20) and (21) as

$$cP_{\varphi,k} = cP^*_{\varphi0,k} + \beta^*_\ell Q^*_{k\ell} + \gamma_0\phi\beta_{0k}\beta_{0m}(cP^*_{\varphi0,m} + \beta^*_\ell Q^*_{m\ell}) + \gamma_0\beta_{0k}(\varphi_0^* + \frac{1}{2}\beta^*_m\beta^*_\ell Q^*_{\ell m}) \tag{22}$$

and

$$H_\varphi = \gamma_0(\varphi_0^* + \frac{1}{2}\beta^*_m\beta^*_\ell Q^*_{\ell m}) + \gamma_0\beta_{0m}(cP^*_{\varphi0,m} + \beta^*_\ell Q^*_{m\ell}) \ , \tag{23}$$

we find that

$$c\vec{P}_\varphi = c\vec{P}^*_\varphi + \gamma_0\phi\vec{\beta}_o(\vec{\beta}_0\vec{P}^*_\varphi) + \gamma_0\vec{\beta}_0 H^*_\varphi \quad + \mathcal{O}(\beta^{*3}) \tag{24}$$

and

$$H_\varphi = \gamma_0 H^*_\varphi + \gamma_0(\vec{\beta}_0\vec{P}^*_\varphi) \quad + \mathcal{O}(\beta^{*3}) \ . \tag{25}$$

Thus, the Lorentz transformation of the FRF energy-momentum 4-vector $(H^*; c\vec{P}^*)$ to the LF energy-momentum 4-vector $(H; c\vec{P})$ is a canonical transformation i.e. the FRF Hamiltonian H^* and its conjugate FRF momentum $\vec{P}^*$ transform into the LF Hamiltonian H and its conjugate LF momentum $\vec{P}$ up to orders of β^{*2}. It is shown in Appendix 7.2 that this holds true for all orders of β^*.

Finally, using the definitions

$$c\vec{P}_{\varphi 0} \equiv \vec{P}_\varphi(\vec{\beta}^* = 0); \qquad H_{\varphi 0} \equiv H_\varphi(\vec{\beta}^* = 0) \ , \tag{26}$$

the total LF Hamiltonian as a function of $t, \vec{x}, \vec{\beta}$ and of $t, \vec{x}, \vec{P}$ is

$$H(t,\vec{x},\vec{\beta}) = H_\mu + H_\varphi = mc^2\gamma + H_{\varphi 0} + \gamma_0\beta_{0,m}\beta^*_\ell Q^*_{\ell m} \quad + \mathcal{O}(\beta^{*2}) \tag{27}$$

$$H(t,\vec{x},\vec{P}) = \sqrt{m^2c^4 + c^2(\vec{P} - \vec{P}_\varphi)^2} + H_{\varphi 0} + \gamma_0\beta_{0,m}\beta^*_\ell Q^*_{\ell m} \quad + \mathcal{O}(\beta^{*2}) \tag{28}$$

and

$$cP_k = mc^2\gamma\beta_k + cP_{\varphi 0,k} + [\delta_{km} + \gamma_0\phi\beta_{0,m}]\,\beta^*_\ell Q^*_{\ell m} \quad + \mathcal{O}(\beta^{*2}) \tag{29}$$

$$c\partial P_k/\partial\beta_n = mc^2\frac{\partial(\gamma\beta_k)}{\partial\beta_n} + \gamma_0\,[\delta_{km} + \gamma_0\phi\beta_{0,m}]\,[\delta_{\ell n} + \gamma_0\phi\beta_{0,n}\beta_{0,\ell}]\,Q^*_{\ell m} \quad + \mathcal{O}(\beta^*) \tag{30}$$

according to eqn. (123).

3 Thomas Effect

The Lorentz boost describes the relation between two frames at constant relative velocity. If however the frames are accelerated, they also rotate with respect to each other by the well-known Thomas rotation [3)] described in Appendix 7.3. While a Lorentz boost from the LF to the TRF by $\vec{\beta}$ or from the LF to the FRF by $\vec{\beta}_0$ does not rotate the 3-dimensional coordinates, a subsequent Lorentz boost from the FRF to the TRF by $\vec{\beta}^*$ results in a rest frame RTRF whose spatial coordinates are rotated with respect to the spatial coordinates in the non-rotated TRF (NRTRF) boosted directly by $\vec{\beta}$ from the LF. In consequence, a physical vector $\vec{V}^{**}$ in the NRTRF appears rotated with respect to its appearence $\tilde{\vec{V}}^{**}$ in the RTRF so that, according to eqn. (150),

$$\vec{V}^{**} = \tilde{\vec{V}}^{**} - \phi[\vec{V}^{**} \times (\vec{\beta}^* \times \vec{\beta}_0)] \quad + \mathcal{O}(\vec{\beta}^{*2}) \ . \tag{31}$$

Therefore, the derivative of $\vec{V}^{**}$ with respect to $\vec{\beta}^*$ is

$$\left[\frac{\partial\vec{V}^{**}}{\partial\beta^*_k}\right]_{\vec{\beta}^*=0} = \left[\frac{\partial\tilde{\vec{V}}^{**}}{\partial\beta^*_k}\right]_{\vec{\beta}^*=0} - \phi[\vec{V}^{**} \times (\hat{k} \times \vec{\beta}_0)] = \left[\frac{\partial\tilde{\vec{V}}^{**}}{\partial\beta^*_k}\right]_{\vec{\beta}^*=0} - \phi[\hat{k}(\vec{\beta}_0\vec{V}^{**}) - \vec{\beta}_0 \tilde{V}^{**}_k] \tag{32}$$

where $\hat{k}$ is the unit vector in k-direction.

This spacial rotation from the RTRF to the NRTRF affects the representation of all vectors in the TRF, in particular those making up the external field φ^{**}. However, if φ^{**} is a function of a scalar or of the scalar product of two vectors $\vec{V}^{**}$ and $\vec{W}^{**}$, it may be expressed in either the rotated or the non-rotated vector since

$$\vec{V}^{**}\vec{W}^{**} = \tilde{\vec{V}}^{**}\tilde{\vec{W}}^{**} - \phi\left[\vec{V}^{**}[\vec{\beta}^*(\vec{\beta}_0\vec{W}^{**}) - \vec{\beta}_0(\vec{\beta}^*\vec{W}^{**})] + \vec{W}^{**}[\vec{\beta}^*(\vec{\beta}_0\vec{V}^{**}) - \vec{\beta}_0(\vec{\beta}^*\vec{V}^{**})]\right] \tag{33}$$

where the expression in the square parentheses is zero.

4 External Forces

The force $\vec{F}$ exerted on a particle by an external field is defined as the total derivative of the motion-related momentum $\vec{P}_\mu$ with respect to time. Since the total time derivative of the canonical momentum is given by the Lagrange equation [1)]

$$\frac{dP_k}{dt} = \left[\frac{\partial L}{\partial x_k}\right]_{\vec{\beta},t} = \frac{dP_{\mu,k}}{dt} + \frac{dP_{\varphi,k}}{dt} \quad , \tag{34}$$

we find

$$F_k \equiv \frac{dP_{\mu,k}}{dt} = \left[\frac{\partial L}{\partial x_k}\right]_{\vec{\beta},t} - \frac{dP_{\varphi,k}}{dt} = \left[\frac{\partial L}{\partial x_k}\right]_{\vec{\beta},t} - \left[\frac{\partial P_{\varphi,k}}{\partial t}\right]_{\vec{x}\vec{\beta}} - \left[\frac{\partial P_{\varphi,k}}{\partial x_m}\right]_{t\vec{\beta}} \frac{dx_m}{dt} - \left[\frac{\partial P_{\varphi,k}}{\partial \beta_m}\right]_{t\vec{x}} \frac{d\beta_m}{dt} \quad . \tag{35}$$

Using eqn. (13), we obtain

$$F_k = -\left[\frac{\partial H_\varphi}{\partial x_k}\right]_{t\vec{\beta}} + c\beta_m \left[\frac{\partial P_{\varphi,m}}{\partial x_k}\right]_{t\vec{\beta}} - c\beta_m \left[\frac{\partial P_{\varphi,k}}{\partial x_m}\right]_{t\vec{\beta}} - \left[\frac{\partial P_{\varphi,k}}{\partial t}\right]_{\vec{x}\vec{\beta}} - \left[\frac{\partial P_{\varphi,k}}{\partial \beta_m}\right]_{t\vec{x}} \frac{d\beta_m}{dt} \tag{36}$$

or

$$\vec{F} = c\vec{\beta} \times (\vec{\nabla} \times \vec{P}_\varphi) - \left[\frac{\partial \vec{P}_\varphi}{\partial t}\right]_{\vec{x}\vec{\beta}} - \vec{\nabla} H_\varphi - \left[\frac{\partial \vec{P}_\varphi}{\partial \beta_m}\right]_{t\vec{x}} \frac{d\beta_m}{dt} = mc\frac{d(\gamma\vec{\beta})}{dt} \quad . \tag{37}$$

where

$$c\partial P_{\varphi,k}/\partial\beta_n = \gamma_0 \left[\delta_{km} + \gamma_0\phi\beta_{0,k}\beta_{0,m}\right]\left[\delta_{\ell n} + \gamma_0\phi\beta_{0,n}\beta_{0,\ell}\right] Q^*_{\ell m} \tag{38}$$

It may be noteworthy that, if a particle traverses a localized force field whose value vanishes outside a given boundary, its momentum $\vec{P}_\mu$ is changed only by the $\vec{\nabla}L$-term since $d\vec{P}_\varphi/dt$ is a total time differential whose integral across the force field is zero.

5 Examples

5.1 Electro-Magnetic Potential

The TRF interaction energy for a particle with a charge e in an electro-magnetic field potential Φ^{**} is $\varphi^{**} = e\Phi^{**}$ and the field part of the FRF Lagrangian is

$$L^*_{em} = -e\Phi^{**}/\gamma^* = -e\Phi^*_0(t^*, \vec{x}^*) + ec\vec{\beta}^*\vec{A}^*_0(t^*, \vec{x}^*) \tag{39}$$

and all quatities $Q^*_{\ell m}$ and higher derivatives with respect to β^* are zero since $(\Phi^*; c\vec{A}^*)$ is a 4-vector and thus

$$\Phi^*(t^*, \vec{x}^*, \vec{\beta}^*) = \gamma^*\Phi^*_0(t^*, \vec{x}^*) - c\gamma^*\vec{\beta}^* \cdot \vec{A}^*_0(t^*, \vec{x}^*) \tag{40}$$

where $\vec{A}^*$ is the electro-magnetic vector potential in the FRF and $\Phi^*_0 \equiv \Phi^*(\vec{\beta}^* = 0)$ and $\vec{A}^*_0 \equiv \vec{A}^*(\vec{\beta}^* = 0)$. The field part $\vec{P}^*_{em}$ of the canonical momentum is therefore

$$P^*_{em,k} = \frac{1}{c}\left[\frac{\partial L^*_{em}}{\partial \beta^*_k}\right] = eA^*_{0,k} \tag{41}$$

and the field part of the RF Hamiltonian is $H^*_{em} = e\Phi^*_0$.

In the LF, the field part L_{em} of the Lagrangian is, according to eqns. (19) and (41),

$$L_{em} = e\left[\gamma_0(\vec{\beta}\vec{\beta}_0 - 1)\Phi^*_0 + c\vec{A}^*_0\vec{\beta} + \gamma_0\phi(\vec{\beta}_0\vec{\beta})(c\vec{A}^*_0\vec{\beta}_0) - \gamma_0(c\vec{A}^*_0\vec{\beta}_0)\right] \equiv -e\Phi/\gamma \qquad (42)$$

and the field parts H_{em} and $\vec{P}_{em}$ of the Hamiltonian and the canonical momentum are

$$H_{em} = e\gamma_0\Phi^*_0 + ec\gamma_0\vec{\beta}_0 \cdot \vec{A}^* \equiv e\Phi_0(t, \vec{x}) \quad ; \qquad (43)$$

$$\vec{P}_{em} = e\left[\vec{A}^* + \phi\gamma_0\vec{\beta}_0(\vec{\beta}_0\vec{A}^*) + \gamma_0\vec{\beta}_0\Phi^*_0/c\right] \equiv e\vec{A}(t, \vec{x}) \qquad (44)$$

according to eqns. (25) to (26).

The resulting electro-magnetic force $\vec{F}_{em}$ on the particle is, according to eqn. (37),

$$\vec{F}_{em} = ec\vec{\beta} \times (\vec{\nabla} \times \vec{A}) - e\left[\frac{\partial\vec{A}}{\partial t}\right]_{\vec{x}} - e\vec{\nabla}\Phi = e(c\vec{\beta} \times \vec{B}) + e\vec{E} \qquad (45)$$

which is the Lorentz force on a particle of charge e in an electro-magnetic field $(\vec{E}; \vec{B})$.

5.2 Stern-Gerlach-Thomas Spin Interaction

The spin $\vec{s}$ of a particle does not change in magnitude and therefore does not change at all in a Lorentz boost from the LF to the NRTRF so that the NRTRF spin vector $\vec{s}^{**} = \vec{s}$. In the FRF however, the spin vector $\tilde{\vec{s}}^*$, where $\tilde{\vec{s}}^*(\vec{\beta}^* = 0) = \vec{s}$, is rotating with $\vec{\beta}^*$ according to eqn. (31) so that

$$\vec{s} = \tilde{\vec{s}}^* - \phi\vec{s} \times (\vec{\beta}^* \times \vec{\beta}_0) = \tilde{\vec{s}}^* - \phi\tilde{\vec{s}}^* \times (\vec{\beta}^* \times \vec{\beta}_0) \quad + \mathcal{O}(\beta^{*2}). \qquad (46)$$

Thus the change of $\vec{s}$ with time t^* in the FRF is

$$\frac{d\vec{s}}{dt^*} = \frac{\partial\vec{s}}{\partial t^*} + \frac{\partial\vec{s}}{\partial\vec{\beta}^*_k}\frac{d\vec{\beta}^*_k}{dt^*} \qquad (47)$$

and, since $\partial\tilde{\vec{s}}^*/\partial\beta^*_k = 0$,

$$\frac{d\vec{s}}{dt^*} = \frac{\partial\vec{s}}{\partial t^*} - \tilde{\vec{s}}^* \times (\frac{d\vec{\beta}^*}{dt^*} \times \vec{\beta}_0)\phi \qquad (48)$$

The first term in eqn. (48) describes the so-called Stern-Gerlach spin precession in the NRTRF electromagnetic field $(\vec{B}^{**}, \vec{E}^{**})$ at fixed $\vec{\beta}^*$ [3]), i.e.

$$\frac{\partial\vec{s}}{\partial t^*} = \vec{\Omega}^{**}_{SG} \times \vec{s} \qquad (49)$$

where, for a particle with charge e, mass m, and a gyro-magnetic factor g,

$$\vec{\Omega}^{**}_{SG}(\vec{x}^*, \vec{\beta}^*, t^*) = -\frac{e}{m}\frac{g}{2}\vec{B}^{**}(\vec{x}^*, \vec{\beta}^*, t^*) \quad . \qquad (50)$$

The second term is related to the Thomas precession $\vec{s}\times\vec{\Omega}^{**}_T$ caused by the acceleration $c(d\vec{\beta}^*/dt^*)$ where

$$\vec{\Omega}^{**}_T(\vec{x^*},\vec{\beta^*},t^*) \equiv \frac{\gamma}{\gamma+1}(\frac{d\vec{\beta}^*}{dt^*}\times\vec{\beta}) \quad . \tag{51}$$

For $\vec{\beta^*}=0$, where the NRTRF becomes the FRF, where $\tilde{\vec{s^*}}=\vec{s}$, $\vec{B}^{**}=\vec{B}^*$,

$$\begin{aligned} \vec{\Omega}^{**}_T(\vec{x^*},\vec{\beta^*}=0,t^*) &= \vec{\Omega}^*_T=\phi(\frac{d\vec{\beta}^*}{dt^*}\times\vec{\beta}_0) \quad , & (52)\\ \vec{\Omega}^{**}_{SG}(\vec{x^*},\vec{\beta^*}=0,t^*) &= \vec{\Omega}^*_{SG}=-\frac{e}{m}\frac{g}{2}\vec{B}^*(\vec{x^*},t^*) \quad , & (53)\end{aligned}$$

the total spin precession in the FRF is

$$\frac{d\vec{s}}{dt^*}=(\vec{\Omega}^*_{SG}+\vec{\Omega}^*_T)\times\vec{s} \tag{54}$$

which, according to ref. 3, results in a NRTRF field interaction energy φ^{**} at $\vec{\beta^*}=0$ of

$$\varphi^{**}(\vec{x^*},\vec{\beta^*}=0,t^*)=\vec{s}\cdot(\vec{\Omega}^*_{SG}+\vec{\Omega}^*_T) \tag{55}$$

where $\vec{s}\cdot\vec{\Omega}^*_{SG}$ is the Stern-Gerlach energy.

If we infinitesimally change the particle velocity by a fixed amount $c\vec{\beta}^*$, the NRTRF, in which the particle velocity is zero and which does not rotate with respect to the FRF, differs infinitesimally from the FRF: The vectors $\vec{\Omega}^*_{SG}$ and $\vec{\Omega}^*_T$ become $\vec{\Omega}^{**}_{SG}$ and $\vec{\Omega}^{**}_T$ and the NRTRF field interaction energy φ^{**} is

$$\varphi^{**}=\vec{s}\cdot(\vec{\Omega}^{**}_{SG}+\vec{\Omega}^{**}_T) \quad . \tag{56}$$

The corresponding field part of the Lagrangian in the NRTRF is then

$$L^{**}_{SGT}=-\vec{s}\cdot(\vec{\Omega}^{**}_{SG}+\vec{\Omega}^{**}_T) \quad . \tag{57}$$

The field part of the FRF Lagrangian L^*_{SGT} is, according to eqn. (3),

$$L^*_{SGT}=-\frac{1}{\gamma^*}\vec{s}\cdot(\vec{\Omega}^{**}_{SG}+\vec{\Omega}^{**}_T) \tag{58}$$

where the precession angular velocities are

$$\vec{\Omega}^{**}_{SG}=-\frac{e}{m}\frac{g}{2}\vec{B}^{**}, \quad \vec{\Omega}^{**}_T=-\frac{\gamma}{\gamma+1}(\vec{\beta}\times d\vec{\beta}^*/dt^*) \tag{59}$$

and where, according to eqn. (31),

$$\vec{B}^{**}=\gamma^*[\vec{B}^*-\vec{\beta}^*\times\vec{E}^*/c-\phi\vec{B}^*\times(\vec{\beta}^*\times\vec{\beta}_0)] \quad +\mathcal{O}(\vec{\beta}^{*2}) \tag{60}$$

and $\vec{B}^*$ is the magnetic field in the FRF at $\vec{\beta^*}=0$.

Thus, the components of $L^*_{SGT} = L^*_{SG} + L^*_T$ are

$$L^*_{SG} = \mu[\vec{s}\vec{B}^* + \vec{\beta}^*\vec{\Sigma}^* + \phi(\vec{\beta}^*\vec{B}^*)(\vec{\beta}_0\vec{s}) - \phi(\vec{\beta}^*\vec{s})(\vec{\beta}_0\vec{B}^*)] \quad + \mathcal{O}(\vec{\beta}^{*2}) \tag{61}$$

and

$$L^*_T = \frac{\gamma}{\gamma^*(\gamma+1)}\vec{s}\cdot(\vec{\beta}\times\frac{d\vec{\beta}^*}{dt^*}) = \frac{\gamma}{\gamma^*(\gamma+1)}\vec{\beta}\cdot(\dot{\vec{\beta}}^*\times\vec{s}) \tag{62}$$

where $\mu \equiv (e/m)(g/2)$, $\dot{\vec{\beta}}^* \equiv \vec{\beta}^*/dt^*$, and $\vec{\Sigma}^* \equiv \vec{s}\times\vec{E}^*/c$.

The resulting field parts of the canonical momentum $\vec{P}^*_{SGT} = \vec{P}^*_{SG} + \vec{P}^*_T$ in the FRF are

$$c\vec{P}^*_{SG} = \mu[\vec{\Sigma}^* + \phi\vec{B}^*(\vec{\beta}_0\vec{s}) - \phi\vec{s}(\vec{\beta}_0\vec{B}^*)] \tag{63}$$

and

$$cP^*_{T,k} = \frac{\hat{k}}{\gamma_0+1}(\dot{\vec{\beta}}^*\times\vec{s}) + \phi\vec{\beta}\cdot(\frac{\partial\dot{\vec{\beta}}^*}{\partial\beta^*_k}\times\vec{s}) \tag{64}$$

according to eqn. (6) and using the relations

$$\frac{\partial(\frac{\gamma}{\gamma+1})}{\partial\beta_m} = \frac{\gamma^3}{(\gamma+1)^2}\beta_m \ ; \qquad \left[\frac{\partial(\vec{\beta}\frac{\gamma}{\gamma+1})}{\partial\beta^*_k}\right]_{\vec{\beta}^*=0} = \frac{\hat{k}}{\gamma_0+1} \quad . \tag{65}$$

The corresponding field part of the FRF Hamiltonian H^*_{SGT} is

$$H^*_{SGT} = c\vec{\beta}^*\vec{P}^*_{SGT} - L^*_{SGT} = -\mu(\vec{s}\vec{B}^*) + \left[\frac{\vec{\beta}^*}{\gamma_0+1} - \frac{\gamma\vec{\beta}}{\gamma^*(\gamma+1)}\right](\dot{\vec{\beta}}^*\times\vec{s}) + \phi\beta^*_k\frac{\partial\dot{\vec{\beta}}^*}{\partial\beta^*_k}(\vec{s}\times\vec{\beta}) \tag{66}$$

or, according to eqns. (118) and (119),

$$H^*_{SGT} = \mu(\vec{s}\vec{B}^*) + \phi\vec{\beta}_0\left[(\beta^*_k\frac{\partial\dot{\vec{\beta}}^*}{\partial\beta^*_k} - \dot{\vec{\beta}}^*)\times\vec{s}\right] = \mu(\vec{s}\vec{B}^*) + \phi[(\beta^*_k\frac{\partial\dot{\vec{\beta}}^*}{\partial\beta^*_k} - \dot{\vec{\beta}}^*)\cdot(\vec{s}\times\vec{\beta}_0] \quad + \mathcal{O}(\vec{\beta}^{*2}) \tag{67}$$

In terms of the velocity-independent LF fields $\vec{B}(t,\vec{x})$ and $\vec{E}(t,\vec{x})$, where

$$\vec{B}^{**} = \gamma\vec{B} - \gamma(\vec{\beta}\times\vec{E}/c) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}\vec{B})\vec{\beta} \quad , \tag{68}$$

$$\vec{E}^{**} = \gamma\vec{E} + \gamma(c\vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}\vec{E})\vec{\beta} \quad , \tag{69}$$

and $\vec{\Sigma} \equiv \vec{s}\times\vec{E}/c$, the field part of the LF Lagrangian $L_{SGT} = (\gamma^*/\gamma)L^*_{SGT}$ is thus [see eqns. (58) and (59)]:

$$L_{SGT} = \mu\left[(\vec{s}\vec{B}) + (\vec{\beta}\vec{\Sigma}) - \frac{\gamma}{\gamma+1}(\vec{s}\vec{\beta})(\vec{B}\vec{\beta})\right] + \frac{1}{\gamma+1}\vec{\beta}(\dot{\vec{\beta}}^*\times\vec{s}) \tag{70}$$

or $L_{SGT} = -\vec{s} \cdot \vec{\Omega}$ where the LF spin precession frequency $\vec{\Omega}$ is

$$\vec{\Omega} = -\mu \left[\vec{B} + (\vec{E}/c \times \vec{\beta}) - \frac{\gamma}{\gamma+1} \vec{\beta}(\vec{B}\vec{\beta}) \right] - \frac{1}{\gamma+1} (\vec{\beta} \times \dot{\vec{\beta}}^*) \tag{71}$$

The corresponding field parts of the LF momenta $P_{SGT,k} = \partial L_{SGT}/\partial \beta_k = P_{SG,k} + P_{T,k}$ are therefore

$$c\vec{P}_{SG} = \mu \left[\vec{\Sigma} - \frac{\gamma^3 \vec{\beta}}{(\gamma+1)^2} (\vec{\beta}\vec{s})(\vec{\beta}\vec{B}) - \frac{\gamma}{\gamma+1} [\vec{s}(\vec{\beta}\vec{B}) + \vec{B}(\vec{\beta}\vec{s})] \right] \tag{72}$$

and

$$c\vec{P}_{T,k} = -\frac{\gamma^3}{(\gamma+1)^2} \beta_k \left[\vec{\beta} \cdot (\dot{\vec{\beta}}^* \times \vec{s}) \right] + \frac{1}{\gamma+1} (\dot{\vec{\beta}}^* \times \vec{s})_k + \frac{1}{\gamma+1} \vec{\beta} \left[\frac{\partial \dot{\vec{\beta}}^*}{\partial \beta_k} \times \vec{s} \right] \tag{73}$$

and the field part of the Hamiltonian $H_{SGT} = c\vec{\beta}\vec{P}_{SGT} - L_{SGT}$ is

$$H_{SGT} = -\mu \left[(\vec{s}\vec{B}) + \frac{\gamma^2}{\gamma+1} (\vec{\beta}\vec{s})(\vec{\beta}\vec{B}) \right] + \frac{1}{\gamma+1} (\vec{s} \times \vec{\beta}) \left[\beta_k \frac{\partial \dot{\vec{\beta}}^*}{\partial \beta_k} - \gamma(\gamma-1) \dot{\vec{\beta}}^* \right] \quad . \tag{74}$$

The corresponding LF force $\vec{F}_{SGT}$ on the particle is, according to eqns. (17) and (35),

$$\vec{F}_{SGT} = \vec{\nabla} L_{SGT} - (\vec{\beta}_0 \vec{\nabla}) c\vec{P}_{SGT} - \dot{\vec{P}}_{SGT} - \dot{\beta}_m [\partial \vec{P}_{SGT}/\partial \beta_m] = mc(\gamma\vec{\beta}) \tag{75}$$

where the dot indicates the partial derivative with respect to time.

The time derivative $d\vec{s}/dt = \vec{\Omega} \times \vec{s}$ does not contribute to $\partial \vec{P}_{SGT}/\partial t$ since

$$-\frac{\partial P_{SGT,k}}{\partial t} = \frac{\partial}{\partial \beta_k} \left[\frac{\partial(\vec{s} \cdot \vec{\Omega})}{\partial t} \right] = \frac{\partial}{\partial \beta_k} \left[\vec{s} \cdot \frac{\partial \vec{\Omega}}{\partial t} + \vec{\Omega} \cdot (\vec{s} \times \vec{\Omega}) \right] = \frac{\partial^2 \vec{\Omega}}{\partial \beta_k \partial t} \cdot \vec{s} \quad , \tag{76}$$

according to eqns. (16) and (72), so that the spin $\vec{s}$ may be considered a constant in the equation (75) for the SGT force.

5.2.1 Example of a Charged Particle Accelerated by the Lorentz Force

A particle with charge Ze in an electro-magnetic field $(\vec{E}, \vec{B})$ is subject to the Lorentz force $Zec\vec{\Lambda}$ where

$$\vec{\Lambda} \equiv \vec{E}/c + \vec{\beta} \times \vec{B} \tag{77}$$

Assuming that the Lorentz force is much larger than the SGT force, i.e.

$$ZecB_{average} \gg \gamma \frac{e}{mc} s\dot{B}_{average} \tag{78}$$

or

$$2.4 \cdot 10^{20} Z \frac{mc^2}{MeV} \gg \gamma \frac{fr}{Hz} \cdot \frac{s}{\hbar} \tag{79}$$

where fr is the oscillating frequency of $\vec{B}$, the resulting acceleration in the TRF is

$$\dot{\vec{\beta}}^* = \frac{\nu}{c} \vec{E}^{**} = \nu(\gamma\vec{\Lambda} - \frac{\gamma^2}{\gamma+1} \vec{\beta} \cdot \epsilon) \tag{80}$$

where $\nu \equiv Ze/m$ and $\epsilon \equiv (\vec{\beta}\vec{E})/c$.

From eqns. (70), (77), and (80), we thus obtain the well-known [4)] LF Lagrangian for the spin motion of a particle of charge Ze, mass m, and gyro-magnetic factor g in an electro-magnetic field $(\vec{E},\vec{B})$:

$$L_{SGT} = \mu\left[(\vec{s}\vec{B}) + (\vec{\beta}\vec{\Sigma}) - \frac{\gamma}{\gamma+1}(\vec{s}\vec{\beta})(\vec{B}\vec{\beta})\right] + \frac{\nu\gamma}{\gamma+1}\left[\{\vec{\Lambda} - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\vec{E}/c)\}\times\vec{s}\right]\vec{\beta} \quad (81)$$

$$= \left[\mu - \nu\frac{\gamma-1}{\gamma}\right](\vec{s}\vec{B}) + \left[\mu - \nu\frac{\gamma}{\gamma+1}\right](\vec{\beta}\vec{\Sigma}) - (\mu-\nu)\frac{\gamma}{\gamma+1}(\vec{\beta}\vec{s})(\vec{\beta}\vec{B}) \quad . \quad (82)$$

In the LF, the particle momentum change is

$$d(mc\gamma\vec{\beta})/dt = Zec\vec{\Lambda}; \qquad d(\gamma\vec{\beta})/dt = \gamma^3(\vec{\beta}\dot{\vec{\beta}})\vec{\beta} + \gamma\dot{\vec{\beta}} = \nu\vec{\Lambda} \quad (83)$$

where $\dot{\vec{\beta}} \equiv \vec{\beta}/dt$. Thus

$$(\vec{\beta}\dot{\vec{\beta}}) = \frac{\nu(\vec{\Lambda}\vec{\beta})}{\gamma^3} = \frac{\nu\epsilon}{\gamma^3}; \qquad \dot{\vec{\beta}} = \frac{\nu}{\gamma}(\vec{\Lambda} - \epsilon\vec{\beta}) \quad . \quad (84)$$

According to eqn. (75), the Stern-Gerlach part of $\vec{F}_{SGT}$ is

$$\vec{F}_{SG} = \vec{\nabla}L_{SG} - c(\vec{\beta}\vec{\nabla})\vec{P}_{SG} - \dot{\vec{P}}_{SG} - \dot{\beta}_m\frac{\partial\vec{P}_{SG}}{\partial\beta_m} \quad . \quad (85)$$

From eqns. (70) and (72), we obtain

$$\frac{1}{\mu}\vec{\nabla}L_{SG} = \vec{\nabla}(\vec{s}\vec{B}) + \vec{\nabla}(\vec{\beta}\vec{\Sigma}) - \frac{\gamma}{\gamma+1}(\vec{\beta}\vec{s})\vec{\nabla}(\vec{\beta}\vec{B}); \quad (86)$$

$$\frac{c}{\mu}(\vec{\beta}\vec{\nabla})\vec{P}_{SG} = (\vec{\beta}\vec{\nabla})\vec{\Sigma} - \frac{\gamma^3\vec{\beta}}{(\gamma+1)^2}(\vec{\beta}\vec{s})(\vec{\beta}\vec{\nabla})(\vec{\beta}\vec{B}) - \frac{\gamma}{\gamma+1}[\vec{s}(\vec{\beta}\vec{\nabla})(\vec{\beta}\vec{B}) + (\vec{\beta}\vec{s})(\vec{\beta}\vec{\nabla})\vec{B}] \quad (87)$$

$$\frac{c}{\mu}\dot{\vec{P}}_{SG} = \dot{\vec{\Sigma}} - \frac{\gamma^3\vec{\beta}}{(\gamma+1)^2}(\vec{\beta}\vec{s})(\vec{\beta}\dot{\vec{B}}) - \frac{\gamma}{\gamma+1}[\vec{s}(\vec{\beta}\dot{\vec{B}}) + (\vec{\beta}\vec{s})\dot{\vec{B}}] \quad (88)$$

and, using the relations

$$\dot{\beta}_m \cdot \partial\gamma/\partial\beta_m = \gamma^3(\vec{\beta}\dot{\vec{\beta}}) = \nu\epsilon \ ; \quad \dot{\beta}_m \cdot \partial\vec{\beta}/\partial\beta_m = \dot{\vec{\beta}} = (\nu/\gamma)(\vec{\Lambda} - \epsilon\vec{\beta}) \ , \quad (89)$$

we find

$$c\frac{\dot{\beta}_m}{\mu\nu}\frac{\partial\vec{P}_{SG}}{\partial\beta_m} = \begin{array}{c} \frac{2\gamma^3}{(\gamma+1)^3}\epsilon\vec{\beta}(\vec{\beta}\vec{s})(\vec{\beta}\vec{B}) - \frac{\gamma^2}{(\gamma+1)^2}\left[\vec{\Lambda}(\vec{\beta}\vec{s})(\vec{\beta}\vec{B}) + \vec{\beta}[(\vec{\Lambda}\vec{s})(\vec{\beta}\vec{B}) + (\vec{\beta}\vec{s})(\vec{\Lambda}\vec{B})]\right] - \\ -\frac{1}{\gamma+1}[\vec{s}(\vec{\Lambda}\vec{B}) + \vec{B}(\vec{\Lambda}\vec{s})] + \frac{\gamma}{(\gamma+1)^2}\epsilon[\vec{s}(\vec{\beta}\vec{B}) + \vec{B}(\vec{\beta}\vec{s})] \end{array} \quad . \quad (90)$$

If, for simplicity, we assume that $\vec{\beta}$ point in the x-3 direction, i.e. $\vec{\beta} = (0;0;\beta)$, we obtain the force parallel to $\vec{\beta}$,

$$\frac{1}{\mu}\vec{F}_{SG,\parallel} = \begin{array}{c} \vec{\nabla}_\parallel(\vec{s}\vec{B}) + (\gamma-1)\vec{\nabla}_\parallel s_3B_3 - \frac{1}{c}\dot{\vec{\Sigma}}_\parallel + \frac{1}{c}\gamma\vec{\beta}s_3\dot{B}_3 + \\ +\frac{\nu}{c}\left[-\frac{\gamma-1}{\gamma+1}\frac{\vec{E}_\parallel}{c}s_3B_3 + \frac{\gamma}{\gamma+1}[\vec{s}_\parallel(\vec{\Lambda}\vec{B}) + \vec{B}_\parallel(\vec{\Lambda}\vec{s})\right] \end{array} \quad . \quad (91)$$

and the force perpedicular to $\vec{\beta}$

$$\frac{1}{\mu}\vec{F}_{SG,\perp} = \vec{\nabla}_\perp[(\vec{s}\vec{B}) - \frac{\gamma-1}{\gamma}s_3B_3] + \beta(\vec{\nabla}_\perp\Sigma_3 - \nabla_3\vec{\Sigma}_\perp) - \frac{1}{c}\dot{\vec{\Sigma}}_\perp + \frac{\gamma-1}{\gamma}\nabla_3(\vec{s}_\perp B_3 + s_3\vec{B}_\perp) + \\ + \frac{\gamma\beta}{(\gamma+1)c}(\vec{s}_\perp\dot{B}_3 + s_3\dot{\vec{B}}_\perp) + \frac{\nu}{c}\left[\frac{\gamma-1}{\gamma+1}\vec{\Lambda}_\perp s_3B_3 + \frac{1}{\gamma+1}[\vec{s}_\perp(\vec{\Lambda}\vec{B}) + \vec{B}_\perp(\vec{\Lambda}\vec{s})] - \frac{\gamma-1}{\gamma(\gamma+1)}E_3[\vec{s}_\perp B_3 + \vec{B}_\perp s_3]\right] \tag{92}$$

The Thomas rotation part of the SGT force is

$$\vec{F}_T = \vec{\nabla}L_T - c(\vec{\beta}\vec{\nabla})\vec{P}_T - \dot{\vec{P}}_T - \dot{\beta}_m\frac{\partial\vec{P}_T}{\partial\beta_m} \quad . \tag{93}$$

From eqns. (70) and (73), we find

$$\frac{1}{\nu}\vec{\nabla}L_T = \frac{\gamma}{\gamma+1}\vec{\nabla}[\vec{\beta}(\vec{\Lambda}\times\vec{s})] = \frac{\gamma}{\gamma+1}\vec{\nabla}[(\vec{\beta}\vec{B})(\vec{\beta}\vec{s}) - \beta^2(\vec{s}\vec{B}) - \vec{\beta}\vec{\Sigma}] \ ; \tag{94}$$

$$\frac{c}{\nu}(\vec{\beta}\vec{\nabla})\vec{P}_T = \frac{\gamma^3}{(\gamma+1)^2}\vec{\beta}(\vec{\beta}\vec{\nabla})(\vec{\Lambda}\times\vec{s})\cdot\vec{\beta} + \frac{\gamma}{\gamma+1}\left[(\vec{\beta}\vec{\nabla})\vec{\Lambda}\times\vec{s} + [(\vec{\beta}\vec{\nabla})\vec{B}\times(\vec{s}\times\vec{\beta})]\right] \tag{95}$$

$$\frac{c}{\nu}\dot{\vec{P}}_T = \frac{\gamma^3}{(\gamma+1)^2}\vec{\beta}[(\dot{\vec{\Lambda}}\times\vec{s})\cdot\vec{\beta}] + \frac{\gamma}{\gamma+1}\left[(\dot{\vec{\Lambda}}\times\vec{s}) + [\dot{\vec{B}}\times(\vec{s}\times\vec{\beta})]\right] \ ; \tag{96}$$

and

$$c\frac{\dot{\beta}_m}{\nu^2}\frac{\partial\vec{P}_T}{\partial\beta_m} = \begin{array}{c} -\frac{\gamma^2(\gamma-1)}{(\gamma+1)^3}\epsilon\vec{\beta}[\vec{\beta}(\vec{\Lambda}\times\vec{s})] + \frac{\gamma^2}{(\gamma+1)^2}\vec{\Lambda}[\vec{\beta}(\vec{\Lambda}\times\vec{s})] + \frac{\epsilon}{(\gamma+1)^2}(\vec{\Lambda}\times\vec{s}) + \\ + \frac{\gamma^2}{(\gamma+1)^2}\vec{\beta}[\vec{\beta}(\vec{\Lambda}\times\vec{s})] + \frac{1}{\gamma+1}\left[[(\vec{\Lambda}\times\vec{B})\times\vec{s}] + [\vec{B}\times(\vec{s}\times\vec{\Lambda})]\right] - \\ -\frac{\epsilon\gamma^2}{(\gamma+1)^2}\vec{\beta}\{[\vec{\beta}\times(\vec{\beta}\times\vec{B})]\cdot\vec{s}\} - \frac{\epsilon}{\gamma+1}[(\vec{\beta}\times\vec{B})\times\vec{s}] - \frac{\epsilon\gamma}{(\gamma+1)^2}[\vec{B}\times(\vec{s}\times\vec{\beta})] \ . \end{array} \tag{97}$$

For $\vec{\beta} = (0;0;\beta)$, the Thomas forces become

$$\frac{1}{\nu}\vec{F}_{T,\parallel} = \begin{array}{c} \frac{\gamma(\gamma-1)}{\gamma+1}\beta\nabla_3\vec{\Sigma}_\parallel + \frac{\gamma^2}{(\gamma+1)c}\dot{\vec{\Sigma}}_\parallel + (\gamma-1)\vec{\nabla}_\parallel(\vec{s}_\perp\vec{B}_\perp) + \gamma\vec{\beta}(\vec{s}_\perp\dot{\vec{B}}_\perp) + \\ + \frac{\nu}{c}\left[\frac{2\epsilon\gamma}{(\gamma+1)^2}\vec{\Sigma}_\parallel + \frac{\vec{E}_\parallel}{c}[(\vec{s}\vec{B}) - \frac{\gamma-1}{\gamma+1}(\vec{s}_\perp\vec{B}_\perp)] - \frac{\gamma}{\gamma+1}\vec{B}_\parallel(\vec{\Lambda}\vec{s}) - \frac{1}{\gamma+1}\vec{s}_\parallel(\vec{\Lambda}\vec{B})\right] \end{array} \tag{98}$$

and

$$\frac{1}{\nu}\vec{F}_{T,\perp} = \frac{\gamma}{\gamma+1}\left[\beta(\nabla_3\vec{\Sigma}_\perp - \vec{\nabla}_\perp\Sigma_3) + \frac{1}{c}\dot{\vec{\Sigma}}_\perp - \frac{\beta}{c}(s_3\dot{\vec{B}}_\perp + \vec{s}_\perp\dot{B}_3)\right] - \frac{\gamma-1}{\gamma}[\vec{\nabla}_\perp(\vec{s}_\perp\vec{B}_\perp) + \nabla_3(s_3\vec{B}_\perp + \vec{s}_\perp B_3)] + \\ + \frac{\nu}{c}\left[\frac{\gamma^2\beta}{(\gamma+1)^2}\vec{\Lambda}_\perp\Sigma_3 + \frac{\epsilon}{(\gamma+1)^2}\vec{\Sigma}_\perp + \vec{\Lambda}_\perp[(\vec{s}\vec{B}) - \frac{\gamma-1}{\gamma+1}s_3B_3] + \frac{\gamma-1}{\gamma(\gamma+1)}\frac{E_3}{c}(s_3\vec{B}_\perp + \vec{s}_\perp B_3)] - \frac{1}{\gamma+1}[\vec{B}_\perp(\vec{s}\vec{\Lambda}) + \vec{s}_\perp(\vec{B}\vec{\Lambda})]\right] \tag{99}$$

The total SGT force on the spin therefore is

$$\vec{F}_{SGT,\parallel} = \begin{array}{c} \nu\frac{\gamma(\gamma-1)}{\gamma+1}\beta\nabla_3\vec{\Sigma}_\parallel + [\nu\frac{\gamma^2}{\gamma+1} - \mu]\frac{1}{c}\dot{\vec{\Sigma}}_\parallel + \nu(\gamma-1)\vec{\nabla}_\parallel(\vec{s}\vec{B}) + (\mu-\nu)(\gamma-1)\vec{\nabla}_\parallel s_3B_3 + \\ + \mu\vec{\nabla}_\parallel(\vec{s}\vec{B}) + \nu\frac{\gamma\vec{\beta}}{c}(\vec{s}\dot{\vec{B}}) + \frac{\mu-\nu}{c}\gamma\vec{\beta}(s_3\dot{B}_3) + \\ + \frac{\nu}{c}\left[\nu\frac{2\gamma\epsilon}{(\gamma+1)^2}\vec{\Sigma}_\parallel + \nu\frac{2}{\gamma+1}\frac{\vec{E}_\parallel}{c}(\vec{s}\vec{B}) + \frac{\mu-\nu}{\gamma+1}\{\gamma\vec{B}_\parallel(\vec{s}\vec{\Lambda}) - (\gamma-1)\frac{\vec{E}_\parallel}{c}(s_3B_3)\} + \frac{\mu\gamma-\nu}{\gamma+1}\vec{s}_\parallel(\vec{B}\vec{\Lambda})\right] \end{array} \tag{100}$$

and

$$\vec{F}_{SGT,\perp} = \begin{array}{c} (\mu-\nu\frac{\gamma}{\gamma+1})[\beta\vec{\nabla}_\perp\Sigma_3 - \beta\nabla_3\vec{\Sigma}_\perp - \frac{1}{c}\dot{\vec{\Sigma}}_\perp] + (\mu-\nu\frac{\gamma-1}{\gamma})\vec{\nabla}_\perp(\vec{s}\vec{B})+ \\ +(\mu-\nu)\left[\frac{\gamma-1}{\gamma}\{\nabla_3(\vec{s}_\perp B_3 + s_3\vec{B}_\perp) - \vec{\nabla}_\perp s_3 B_3\} + \frac{\gamma\beta}{(\gamma+1)c}[\vec{s}_\perp \dot{B}_3 + s_3\dot{\vec{B}}_\perp]\right]+ \\ +\frac{\nu^2}{c}\left[\vec{\Lambda}_\perp\frac{\gamma^2\beta}{(\gamma+1)^2}\Sigma_3 + \frac{\epsilon}{(\gamma+1)^2}\vec{\Sigma}_\perp + \vec{\Lambda}_\perp(\vec{s}\vec{B})\right]+ \\ +\frac{\nu(\mu-\nu)}{c}\left[\frac{1}{\gamma+1}[\vec{s}_\perp(\vec{\Lambda}\vec{B}) + \vec{B}_\perp(\vec{\Lambda}\vec{s})] + \frac{\gamma-1}{\gamma+1}[\vec{\Lambda}_\perp s_3 B_3 - \frac{E_3}{c\gamma}(\vec{s}_\perp B_3 + s_3\vec{B}_\perp)]\right] \; . \end{array} \tag{101}$$

For large values of γ, these forces approach

$$\vec{F}_{SGT,\parallel} \approx \begin{array}{c} \nu\gamma\left[\nabla_3\vec{\Sigma}_\parallel + \frac{1}{c}\dot{\vec{\Sigma}}_\parallel + \vec{\nabla}_\parallel(\vec{s}\vec{B}) + \frac{\vec{\beta}}{c}(\vec{s}\dot{\vec{B}}) + (\mu-\nu)\{\vec{\nabla}_\parallel s_3 B_3 + \frac{\vec{\beta}}{c}s_3\dot{B}_3\}\right] - \\ -2\nu\nabla_3\vec{\Sigma}_\parallel - \frac{\nu-\mu}{c}\dot{\vec{\Sigma}}_\parallel + (\mu-\nu)\vec{\nabla}_\parallel(\vec{s}\vec{B}) - (\mu-\nu)\vec{\nabla}_\parallel(s_3B_3)+ \\ +\frac{\nu}{c}(\mu-\nu)\left[\vec{B}_\parallel(\vec{s}\vec{\Lambda}) - \frac{\vec{E}_\parallel}{c}s_3B_3\right] + \frac{\nu\mu}{c}\vec{s}_\parallel(\vec{B}\vec{\Lambda}) \end{array} \tag{102}$$

or

$$\vec{F}_{SGT,\parallel} \approx \begin{array}{c} \gamma\vec{\beta}\frac{d}{dt}\left[\nu\Sigma_\parallel + \nu(\vec{s}\vec{B}) + (\mu-\nu)s_3B_3\right] - 2\nu\nabla_3\vec{\Sigma}_\parallel + (\mu-\nu)[\vec{\nabla}_\parallel(\vec{s}_\perp\vec{B}_\perp) - \frac{1}{c}\dot{\vec{\Sigma}}_\parallel]+ \\ +\frac{\nu}{c}(\mu-\nu)\left[\vec{B}_\parallel(\vec{s}\vec{\Lambda}) - \frac{\vec{E}_\parallel}{c}s_3B_3\right] + \frac{\nu\mu}{c}\vec{s}_\parallel(\vec{B}\vec{\Lambda}) \; . \end{array} \tag{103}$$

and

$$\vec{F}_{SGT,\perp} \approx \begin{array}{c} (\mu-\nu)\left[\vec{\nabla}_\perp\Sigma_3 - \nabla_3\vec{\Sigma}_\perp - \frac{1}{c}\dot{\vec{\Sigma}}_\perp + \vec{\nabla}_\perp(\vec{s}\vec{B}) + \nabla_3(\vec{s}_\perp B_3 + s_3\vec{B}_\perp) - \vec{\nabla}_\perp(s_3B_3)\right]+ \\ +\frac{\mu-\nu}{c}(\vec{s}_\perp\dot{B}_3 + s_3\dot{\vec{B}}_\perp) + \frac{\nu^2}{c}\vec{\Lambda}_\perp(\Sigma_3 + \vec{s}\vec{B}) + \frac{\nu}{c}(\mu-\nu)\vec{\Lambda}_\perp s_3B_3 = \\ = (\mu-\nu)\left[\vec{\nabla}_\perp(\Sigma_3 + \vec{s}_\perp\vec{B}_\perp) + \frac{d}{dt}(\vec{s}_\perp B_3 + s_3\vec{B}_\perp - \vec{\Sigma}_\perp)\right] + \frac{\nu}{c}\vec{\Lambda}_\perp\left[\nu\Sigma_3 + \nu(\vec{s}_\perp\vec{B}_\perp) + \mu s_3B_3\right] \end{array} \tag{104}$$

In the FRF, $\vec{F}^*_{SGT}$ is obtained by letting $\vec{\beta}$ approach zero:

$$\vec{F}^*_{SGT} = \left[\frac{\nu}{2} - \mu\right]\frac{1}{c}\dot{\vec{\Sigma}}^* + \mu\vec{\nabla}^*(\vec{s}\vec{B}^*) + \frac{\nu^2}{c^2}\vec{E}^*(\vec{s}\vec{B}^*) + \frac{\nu}{2c^2}(\mu-\nu)[\vec{B}^*(\vec{s}\vec{E}^*) + \vec{s}(\vec{B}^*\vec{E}^*)] \tag{105}$$

since $\vec{\Lambda}^* = \vec{E}^*/c$; $\epsilon = 0$; $\gamma = 1$.

The SGT-force is composed of two parts, one proportional to ν and μ (which are of the same order of magnitude) which depends on space and time variations of the electromagnetic field, and another proportional to ν^2 and $\nu\mu$ depending on the field itself. Since $\nu = Ze/m$ for charged particles is largest for electrons, the ratio of the two parts of the force is larger by more than two orders of magnitude than for the next heavier charged particle, the muon. At the same time, $\mu - \nu$ is about three orders of magnitude smaller than ν or μ for electrons and muons. It is useful therefore to list the components of the SGT-force for electrons and muons neglecting the $(\mu - \nu)$-terms:

$$\vec{F}_{SGT,\parallel;el.} \approx \begin{array}{c} \nu\left[\frac{\gamma(\gamma-1)}{\gamma+1}\beta\nabla_\parallel\vec{\Sigma}_\parallel + \frac{\gamma^2-\gamma-1}{(\gamma+1)c}\dot{\vec{\Sigma}}_\parallel + \gamma\vec{\nabla}_\parallel(\vec{s}\vec{B}) + \frac{\gamma\vec{\beta}}{c}(\vec{s}\dot{\vec{B}})\right]+ \\ +\frac{\nu^2}{c}\left[\frac{2\gamma\epsilon}{(\gamma+1)^2}\vec{\Sigma}_\parallel + \frac{2}{(\gamma+1)^2}\frac{\vec{E}_\parallel}{c}(\vec{s}\vec{B}) + \frac{\gamma-1}{\gamma+1}\vec{s}_\parallel(\vec{B}\vec{\Lambda})\right] \end{array} \tag{106}$$

and

$$\vec{F}_{SGT,\perp;el.} \approx \begin{array}{c} \nu\left[\frac{\beta}{\gamma+1}[\vec{\nabla}_\perp\Sigma_3 - \nabla_3\vec{\Sigma}_\perp] - \frac{1}{(\gamma+1)c}\dot{\vec{\Sigma}}_\perp + \frac{1}{\gamma}\vec{\nabla}_\perp(\vec{s}\vec{B})\right] + \\ +\frac{\nu^2}{c}\left[\vec{\Lambda}_\perp\{\frac{\gamma^2\beta}{(\gamma+1)^2}\Sigma_3 + (\vec{s}\vec{B})\} + \frac{\epsilon}{(\gamma+1)^2}\vec{\Sigma}_\perp\right] \end{array} \quad (107)$$

where, for $\gamma \gg 1$,

$$\vec{F}_{SGT,\parallel;el.} \approx \nu\gamma\left[\nabla_3\vec{\Sigma}_\parallel + \frac{1}{c}\dot{\vec{\Sigma}}_\parallel + \vec{\nabla}_\parallel(\vec{s}\vec{B}) + \frac{1}{c}(\dot{\vec{s}\vec{B}})\right] - 2\nu[\nabla_3\vec{\Sigma}_\parallel + \frac{1}{c}\dot{\vec{\Sigma}}_\parallel] + \frac{\nu^2}{c}\vec{s}_\parallel(\vec{B}\vec{\Lambda}) \quad (108)$$

and

$$\vec{F}_{SGT,\perp;el.} \approx \frac{\nu^2}{c}\vec{\Lambda}_\perp[\Sigma_3 + (\vec{s}\vec{B})] \quad , \quad (109)$$

and where the FRF value of $\vec{F}^*_{SGT}$ is

$$\vec{F}^*_{SGT} = -\frac{\nu}{2c}\dot{\vec{\Sigma}}^* + \nu\vec{\nabla}^*(\vec{s}\vec{B}^*) + \frac{\nu^2}{c^2}\vec{E}^*(\vec{s}\vec{B}^*) \quad . \quad (110)$$

For a particle traversing a localized force field as mentioned at the end of chapter 4, the net SGT force $\vec{F}_{SGT,net} = \vec{\nabla}L_{SGT}$ affecting the total momentum change integrated over the traversal is, according to eqns. (81), (86), and (94),

$$\vec{F}_{SGT,net} = \left[\mu - \nu\frac{\gamma-1}{\gamma}\right]\vec{\nabla}(\vec{s}\vec{B}) + \left[\mu - \nu\frac{\gamma}{\gamma+1}\right]\vec{\nabla}(\vec{\beta}\vec{\Sigma}) - \frac{\gamma}{\gamma+1}(\mu-\nu)(\vec{\beta}\vec{s})\vec{\nabla}(\vec{\beta}\vec{B}) \quad (111)$$

which, for large γ, becomes approximately proportional to $\mu - \nu$ and is therefore suppressed by $1/\gamma$ for electrons down to a minimal value comparable to the value for protons.

Furthermore, the energy change dE^2 imparted by the SGT-force to an incident particle of energy $E = mc^2\gamma$ and momentum $\vec{p} = mc\gamma\vec{\beta}$ is

$$dE^2 = 2E \cdot dE = 2c^2\vec{p} \cdot d\vec{p} = 2mc^2\gamma c\vec{\beta} \cdot \vec{\nabla}L_{SGT} \cdot dt \; . \quad (112)$$

Since the time integral of $dL_{SGT}/dt = (\partial L_{SGT}/\partial t)dt + (\vec{\nabla}L_{SGT} \cdot c\vec{\beta})dt$ over the particle path through a localized field is zero, we find that the total energy change ΔE over the path is given by the relation (with E_0 the incident particle energy)

$$\int_{path} dE^2 \approx 2E_0\Delta E = 2mc^2\int_{path} dt\gamma c\vec{\beta} \cdot \vec{\nabla}L_{SGT} \; ; \quad \Delta E \approx -\int_{path} dt(\partial L_{SGT}/\partial t) \quad (113)$$

if the velocity factor γ stays approximately constant, neglecting terms like $(\Delta E)^2$ and $(\Delta\vec{p})^2$ quadratic in L_{SGT}.

6 References

1) P.M. Morse and H. Feshbach; "Methods of Theoretical Physics". McGraw-Hill, 1953, pp. 280-283.
2) ibid.; p. 297.
3) J.D. Jackson; "Classical Electrodynamics", Wiley & Sons, 1975, ch. 11.8.
4) A.A. Pomeranskii and J.B. Khriplovich, Journal of Experimental and Theoretical Physics; Vol. 86, No.5, 1998.

7 Appendix

7.1 Lorentz Transform of the Space-Time 4-Vector and Velocity

Beginning in frame 1 with the infinitesimal space-time 4-vector $(cdt^*; \vec{dx}^*)$ and the velocity $c\vec{\beta}^* = \vec{dx}^*/dt^*$ in this frame, we Lorentz-boost this 4-vector by a velocity $-c\vec{\beta}_0$ into frame 2 to obtain the boosted 4-vector $(cdt; \vec{dx})$ and the velocity $c\vec{\beta} = \vec{dx}/dt$ relative to frame 2, where

$$cdt = \gamma_0 cdt^* + \gamma_0 \vec{\beta}_0 \cdot \vec{dx}^*; \qquad \vec{dx} = \vec{dx}^* + \phi\gamma_0\vec{\beta}_0(\vec{\beta}_0 \cdot \vec{dx}^*) + \gamma_0\vec{\beta}_0 cdt^* \tag{114}$$

so that

$$\vec{\beta} = \frac{\vec{\beta}^* + \frac{\gamma_0^2}{\gamma_0+1}\vec{\beta}_0(\vec{\beta}_0 \cdot \vec{\beta}^*) + \gamma_0\vec{\beta}_0}{\gamma_0(1 + \vec{\beta}_0 \cdot \vec{\beta}^*)}; \qquad \vec{\beta}^* = \frac{\vec{\beta} + \phi\gamma_0\vec{\beta}_0(\vec{\beta}_0 \cdot \vec{\beta}) - \gamma_0\vec{\beta}_0}{\gamma_0(1 - \vec{\beta}_0 \cdot \vec{\beta})} \quad . \tag{115}$$

From these relations between frame-1 and frame-2 velocities, a number of other useful relations may be derived, such as:

$$\vec{\beta}_0 \cdot \vec{\beta} = \frac{\vec{\beta}_0(\vec{\beta}^* + \vec{\beta}_0)}{1 + \vec{\beta}_0 \cdot \vec{\beta}^*}; \qquad \vec{\beta}_0 \cdot \vec{\beta}^* = \frac{\vec{\beta}_0(\vec{\beta} - \vec{\beta}_0)}{1 - \vec{\beta}_0 \cdot \vec{\beta}} \tag{116}$$

and

$$\frac{1}{\gamma^2} = \frac{\gamma_0^2(1+\vec{\beta}_0\vec{\beta}^*)^2 - \vec{\beta}^{*2} - \phi(\gamma_0^2-\gamma_0)(\vec{\beta}_0\vec{\beta}^*)^2 - \gamma_0^2\vec{\beta}_0^{\,2} - 2\phi\gamma_0(\vec{\beta}_0\vec{\beta}^*)^2 - 2\gamma_0\vec{\beta}_0\vec{\beta}^* - 2(\gamma_0^2-\gamma_0)\vec{\beta}_0\vec{\beta}^*}{\gamma_0^2(1+\vec{\beta}_0\vec{\beta}^*)^2} \tag{117}$$

or

$$\begin{aligned} \gamma^2 &= \frac{\gamma_0^2(1+\vec{\beta}_0\vec{\beta}^*)^2}{1 - \vec{\beta}^{*2} + 2\gamma_0[\gamma_0 - 1 - \gamma_0 + 1]\vec{\beta}_0\vec{\beta}^* - \phi[\gamma_0^2 - \gamma_0 + 2\gamma_0 - \gamma_0^2 - \gamma_0](\vec{\beta}_0\vec{\beta}^*)^2} && (118) \\ &= \gamma^{*2} \cdot \gamma_0^2(1+\vec{\beta}_0\vec{\beta}^*)^2 \quad , && (119) \end{aligned}$$

so that

$$\gamma = \gamma^*\gamma_0(1+\vec{\beta}_0\vec{\beta}^*); \quad \gamma^* = \gamma\gamma_0(1-\vec{\beta}_0\vec{\beta}); \quad \gamma_0^2(1+\vec{\beta}_0\vec{\beta}^*)(1-\vec{\beta}_0\vec{\beta}) = 1 \tag{120}$$

and

$$\vec{\beta} = \frac{\gamma^*}{\gamma}\left[\vec{\beta}^* + \phi\gamma_0\vec{\beta}_0(\vec{\beta}_0\vec{\beta}^*) + \gamma_0\vec{\beta}_0\right]; \quad \vec{\beta}^* = \frac{\gamma}{\gamma^*}\left[\vec{\beta} + \phi\gamma_0\vec{\beta}_0(\vec{\beta}_0\vec{\beta}) - \gamma_0\vec{\beta}_0\right] \quad . \tag{121}$$

The partial derivatives of $\vec{\beta}$ with respect to $\vec{\beta}^*$ are

$$\begin{aligned} \frac{\partial\beta_\ell}{\partial\beta_k^*} &= \frac{\gamma^*}{\gamma}\delta_{k\ell} + \frac{\gamma^{*2}\gamma_0}{\gamma^2}\left[\phi\gamma_0\beta_{0\ell}\beta_{0k}(1+\vec{\beta}_0\vec{\beta}^*) - \beta_{0k}[\beta_\ell^* + \phi\gamma_0\beta_{0\ell}(\vec{\beta}_0\vec{\beta}^*) + \gamma_0\beta_{0\ell}]\right] && (122) \\ &= \frac{\gamma^*}{\gamma}\delta_{k\ell} - \frac{\gamma^{*2}}{\gamma^2}\gamma_0\beta_{0k}\left[\phi\beta_{0\ell} + \beta_\ell^*\right] \quad . && (123) \end{aligned}$$

Inversely, by replacing $\vec{\beta}^*$ with $\vec{\beta}$ and $\vec{\beta}_0$ with $-\vec{\beta}_0$ (see eqn. (123)), we find

$$\frac{\partial\beta^*_\ell}{\partial\beta_k} = \frac{\gamma}{\gamma^*}\delta_{k\ell} - \frac{\gamma^2}{\gamma^{*2}}\gamma_0\beta_{0k}\left[\phi\beta_{0\ell} - \beta_\ell\right] = \frac{\gamma}{\gamma^*}[\delta_{k\ell} + \gamma_0\phi\beta_{0k}\beta_{0\ell} + \gamma_0\beta_{0k}\beta^*_\ell] \quad . \tag{124}$$

For $\vec{\beta}^* \longrightarrow 0$ (rest frame), these derivatives become

$$\frac{\partial\beta_\ell}{\partial\beta^*_k} = \frac{\delta_{k\ell}}{\gamma_0} - \frac{\beta_{0k}\beta_{0\ell}}{\gamma_0+1}; \qquad \frac{\partial\beta^*_\ell}{\partial\beta_k} = \gamma_0\delta_{k\ell} + \phi\gamma_0^2\beta_{0k}\beta_{0\ell} \quad . \tag{125}$$

If $\vec{\beta}_0$ points in the x_3-direction ($\vec{\beta}_0 = 0; 0; \beta_0$), we obtain

$$\frac{\partial\beta_\ell}{\partial\beta^*_k} = \frac{\gamma^*}{\gamma}\delta_{k\ell} - \frac{\gamma^{*2}}{\gamma^2}\gamma_0\beta_0\delta_{k3}[\phi\beta_0\delta_{3\ell} + \beta^*_\ell] \tag{126}$$

$$= \frac{\gamma^*}{\gamma}\delta_{k\ell}\left[1 - \frac{\gamma^*}{\gamma}(\gamma_0 - 1)\delta_{k3}\right] - \frac{\gamma^{*2}}{\gamma^2}\gamma_0\beta_0\beta^*_\ell\delta_{k3} \tag{127}$$

which, for $\beta_{0\ell} = \beta_0\delta_{3\ell}$ and $\vec{\beta}^* \longrightarrow 0$, becomes

$$\frac{\partial\beta_\ell}{\partial\beta^*_k} = \frac{\delta_{k\ell}}{\gamma_0}\left[1 - \frac{\gamma_0 - 1}{\gamma_0}\delta_{k3}\right] \quad . \tag{128}$$

7.2 Canonical Property of the Lorentz Transform of the Energy-Momentum 4-Vector $(H^*; c\vec{P}^*)$

The space component $\vec{P}^*$ of the energy-momentum 4-vector $(H^*; c\vec{P}^*)$ defined in Section 2 is the canonically conjugate momentum of the Hamiltonian H^* and is defined by the Hamilton canonical equations [1)]

$$\left[\frac{\partial H^*}{\partial P^*_k}\right]_{\vec{x}^*,t^*} = c\beta^*_k; \qquad \left[\frac{\partial H^*}{\partial x^*_k}\right]_{\vec{P}^*,t^*} = -\frac{dP^*_k}{dt^*} \quad . \tag{129}$$

Alternatively, we may define the associated Lagrangian L^* as

$$L^* \equiv c\vec{\beta}^* \cdot \vec{P}^* - H^* \quad . \tag{130}$$

The partial derivative with respect to β^*_k is

$$\left[\frac{\partial L^*}{\partial\beta^*_k}\right]_{\vec{x}^*,t^*} = cP^*_k + c\beta^*_m\left[\frac{\partial P^*_m}{\partial\beta^*_k}\right]_{\vec{x}^*,t^*} - \left[\frac{\partial H^*}{\partial\beta^*_k}\right]_{\vec{x}^*,t^*} \quad . \tag{131}$$

Since, according to eqn. (129),

$$\left[\frac{\partial H^*}{\partial\beta^*_k}\right]_{\vec{x}^*,t^*} = \left[\frac{\partial H^*}{\partial P^*_m}\right]_{\vec{x}^*,t^*}\left[\frac{\partial P^*_m}{\partial\beta^*_k}\right]_{\vec{x}^*,t^*} + \left[\frac{\partial H^*}{\partial x^*_m}\right]_{\vec{P}^*,t^*}\left[\frac{\partial x^*_m}{\partial\beta^*_k}\right]_{\vec{x}^*,t^*} + \left[\frac{\partial H^*}{\partial t^*}\right]_{\vec{x}^*,\vec{P}^*}\left[\frac{\partial t^*}{\partial\beta^*_k}\right]_{\vec{x}^*,t^*} = \frac{\partial P^*_m}{\partial\beta^*_k}c\beta^*_m \tag{132}$$

so that

$$c\vec{\beta}_m^* \left[\frac{\partial P_m^*}{\partial \beta_k^*}\right]_{\vec{x}^*,t^*} - \left[\frac{\partial H^*}{\partial \beta_k^*}\right]_{\vec{x}^*,t^*} = 0 \quad , \tag{133}$$

we obtain the relation

$$\frac{\partial L^*}{\partial \beta_k^*} = cP_k^* \quad . \tag{134}$$

Furthermore, since

$$\left[\frac{\partial H^*}{\partial x_k^*}\right]_{\vec{\beta}^*,t^*} = \left[\frac{\partial H^*}{\partial x_k^*}\right]_{\vec{P}^*,t^*} + \left[\frac{\partial H^*}{\partial P_m^*}\right]_{\vec{x}^*,t^*} \left[\frac{\partial P_m^*}{\partial x_k^*}\right]_{\vec{\beta}^*,t^*} = -\frac{dP_k^*}{dt} + c\beta_m^* \left[\frac{\partial P_m^*}{\partial x_k^*}\right]_{\vec{\beta}^*,t^*} \quad , \tag{135}$$

we find the Lagrange relation

$$\left[\frac{\partial L^*}{\partial x_m^*}\right]_{\vec{\beta}^*,t^*} = c\beta_m^* \left[\frac{\partial P_m^*}{\partial x_k^*}\right]_{\vec{\beta}^*,t^*} - \left[\frac{\partial H^*}{\partial x_k^*}\right]_{\vec{\beta}^*,t^*} = \frac{dP_k^*}{dt} \quad . \tag{136}$$

Therefore, the Lagrangian relations (134) and (136) are equivalent to the Hamiltonian canonical equations (129).

The Lorentz-boosted Lagrangian of eqn. (16) is, in terms of the frame-1 Hamiltonian H^* and its canonical frame-1 momentum $\vec{P}^*$,

$$L = \frac{\gamma^*}{\gamma} L^* = \frac{\gamma^*}{\gamma} \beta_m^* cP_m^* - \frac{\gamma^*}{\gamma} H^* \quad . \tag{137}$$

The corresponding canonical frame-2 momentum is therefore

$$\frac{\partial L}{\partial \beta_k} = \frac{\partial}{\partial \beta_k}\left(\frac{\gamma^*}{\gamma}\beta_m^*\right) cP_m^* - \frac{\partial}{\partial \beta_k}\left(\frac{\gamma^*}{\gamma}\right) H^* \;+\; \left[\frac{\gamma^*}{\gamma}\beta_m^* \frac{c\partial P_m^*}{\partial \beta_\ell^*} - \frac{\gamma^*}{\gamma}\frac{\partial H^*}{\partial \beta_\ell^*}\right] \frac{\partial \beta_\ell^*}{\partial \beta_k} \tag{138}$$

where the term in square parentheses is zero according to eqn. (132). Therefore, using eqns. (120) and (121), we obtain

$$\frac{\partial L}{\partial \beta_k} = \frac{\partial}{\partial \beta_k}\left[\beta_m + \phi\gamma_0\beta_{0,m}(\vec{\beta}_0\vec{\beta}) - \gamma_0\beta_{0,m}\right] cP_m^* - \frac{\partial}{\partial \beta_k}[\gamma_0(1 - \vec{\beta}\vec{\beta}_0)]H^* \tag{139}$$

or

$$\frac{\partial L}{\partial \beta_k} = cP_k^* + \phi\gamma_0\beta_{0,k}(\vec{\beta}_0 c\vec{P}^*) + \gamma_0\beta_{0,k}H^* = cP_k \quad . \tag{140}$$

The frame-2 Hamiltonian H is, according to eqn. (16),

$$H = \begin{array}{c} \beta_k cP_k^* + \phi\gamma_0(\vec{\beta}_0\vec{\beta})(\vec{\beta}_0 c\vec{P}^*) + \gamma_0(\vec{\beta}_0\vec{\beta})H^* - \\ -c\vec{\beta}\vec{P}^* - \phi\gamma_0(\vec{\beta}_0\vec{\beta})(\vec{\beta}_0 c\vec{P}^*) + \gamma_0(\vec{\beta}_0\vec{P}^*) + \gamma_0(1 - \vec{\beta}\vec{\beta}_0)H^* \end{array} \tag{141}$$

or

$$H = \gamma_0 H^* + \gamma_0(\vec{\beta}_0 c\vec{P}^*) \tag{142}$$

The frame-1 Hamiltonian H^* and its canonically conjugate frame-1 momentum $\vec{P}^*$ transform, according to eqns. (140) and (142), to the Lorentz-boosted frame-2 Hamiltonian H and its canonically conjugate momentum $\vec{P}$ like the components of an energy-momentum 4-vector $(H; c\vec{P})$. The Lorentz boost is therefore a canonical transformation..

7.3 Thomas Effect of an Infinitesimal Velocity on Vectors in a Lorentz-Boosted Rest Frame

A Lorentz boost $A(\vec{\beta_0})$ of a 4-vector v by a velocity $c\vec{\beta_0}$ from the laboratory frame (LF) results in a 4-vector $v^0 = A(\vec{\beta_0})v$. After a further boost $A(\vec{\beta^*})$ by an infinitesimal velocity $c\vec{\beta^*}$, we obtain a 4-vector $\tilde{v}^*$:

$$\tilde{v}^* = A(\vec{\beta^*})A(\vec{\beta_0})v \quad . \tag{143}$$

A direct Lorentz boost of v to a frame with the same velocity relative to the LF as the $\tilde{v}^*$-frame results in a 4-vector v^*:

$$v^* = A(\vec{\beta_0} + \delta\vec{\beta})v \tag{144}$$

with $v_0^* = \tilde{v}_0^*$. The relation between $\tilde{v}^*$ and v^* can be expressed as $\tilde{v}^* = \tilde{A}v^*$ where

$$\tilde{A} = A(\vec{\beta^*})A(\vec{\beta_0})A(-\vec{\beta_0} - \delta\vec{\beta}) \quad . \tag{145}$$

If $\vec{\beta_0}$ points in the x_3-direction, we find

$$A(\vec{\beta_0}) = \begin{pmatrix} \gamma_0 & 0 & 0 & -\gamma_0\beta_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_0\beta_0 & 0 & 0 & \gamma_0 \end{pmatrix} ; \quad A(\vec{\beta^*}) = \begin{pmatrix} 1 & -\beta_1^* & -\beta_2^* & -\beta_3^* \\ -\beta_1^* & 1 & 0 & 0 \\ -\beta_2^* & 0 & 1 & 0 \\ -\beta_3^* & 0 & 0 & 1 \end{pmatrix} \tag{146}$$

neglecting terms of order β_k^{*2}, so that

$$A(\vec{\beta^*})A(\vec{\beta_0}) = \begin{pmatrix} \gamma_0 + \gamma_0\beta_0\beta_3^* & -\beta_1^* & -\beta_2^* & -\gamma_0\beta_0 - \gamma_0\beta_3^* \\ -\gamma_0\beta_1^* & 1 & 0 & \gamma_0\beta_0\beta_1^* \\ -\gamma_0\beta_2^* & 0 & 1 & \gamma_0\beta_0\beta_2^* \\ -\gamma_0\beta_0 - \gamma_0\beta_3^* & 0 & 0 & \gamma_0 + \gamma_0\beta_0\beta_3^* \end{pmatrix} + \mathcal{O}(\vec{\beta}^{*2}) \quad . \tag{147}$$

Since

$$A(\vec{\beta_0} + \delta\vec{\beta}) = \begin{pmatrix} \gamma_0 + \gamma_0^3\beta_0\delta\beta_3 & -\gamma_0\delta\beta_1 & -\gamma_0\delta\beta_2 & -\gamma_0\beta_0 - \gamma_0^3\delta\beta_3 \\ -\gamma_0\delta\beta_1 & 1 & 0 & \phi\gamma_0\beta_0\delta\beta_1 \\ -\gamma_0\delta\beta_2 & 0 & 1 & \phi\gamma_0\beta_0\delta\beta_2 \\ -\gamma_0\beta_0 - \gamma_0^3\delta\beta_3 & 0 & 0 & \gamma_0 + \gamma_0^3\beta_0\delta\beta_3 \end{pmatrix} + \mathcal{O}(\vec{\beta}^{*2}) \quad , \tag{148}$$

we require $\gamma_0\delta\beta_1 = \beta_1^*$; $\gamma_0\delta\beta_2 = \beta_2^*$; $\gamma_0^2\delta\beta_3 = \beta_3^*$ to satisfy the condition $v_0^* = \tilde{v}_0^*$ and obtain

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \phi\beta_0\beta_1^* \\ 0 & 0 & 1 & \phi\beta_0\beta_2^* \\ 0 & -\phi\beta_0\beta_1^* & -\phi\beta_0\beta_2^* & 1 \end{pmatrix} + \mathcal{O}(\vec{\beta}^{*2}) \quad . \tag{149}$$

Therefore, $\tilde{A}$ represents the sum of two rotations around the x_1- and x_2-axes by the infinitesimal angles $\phi\beta_0\beta_2^*$ and $\phi\beta_0\beta_1^*$, respectively, from the vector $\vec{v^*}$ to the vector $\tilde{\vec{v^*}}$. This may be summarized for arbitrary directions of $\vec{\beta_0}$ by the relation

$$\tilde{\vec{v^*}} = \vec{v^*} + \phi\left[\vec{v^*} \times (\vec{\beta^*} \times \vec{\beta_0})\right] \quad + \mathcal{O}(\vec{\beta}^{*2}) \tag{150}$$

which represents a rotation of $\vec{v^*}$ around the $\vec{\beta^*} \times \vec{\beta_0}$ -axis by an infinitesimal angle $\phi \cdot |\vec{\beta^*} \times \vec{\beta_0}|$.