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Talman SPIN2016 comments

The reference is to Talman's talk at SPIN 2016, available here https://indico.cern.ch/event/570680/contributions/2310168/attachments/1341808/2026008/Talman_Spin2016-talk.pdf.

- Talman denotes the lab frame by K and the electron rest frame by K'. But K' (as Talman employs it) is *not* the rest frame. To clarify concepts, I define two terms: "longitudinal" means "parallel to the particle momentum (velocity)" and "axial" means "parallel to the beamline reference axis." I employ a coordinate system (x, y, s). The boost from K to K' is an axial boost, such that $p_s(K') = 0$ (or $v_s(K') = 0$). However the particle still has a nonzero momentum $p_{x,y}(K') = p_{x,y}(K)$. What this means is that there is still a $\boldsymbol{v} \times \boldsymbol{E}$ term for the spin-orbit interaction in the frame K'.

- More generally, for relativistic particles the Stern-Gerlach force is *not* proportional to the magnetic dipole moment. The coupling to the spin is given by the same $\boldsymbol{\Omega} \cdot \boldsymbol{s}$ Hamiltonian which yields the BMT equation.[1] Writing $a = (g-2)/2$, the transverse magnetic fields in $\boldsymbol{\Omega} \cdot \boldsymbol{s}$ are multiplied by the coefficient $(\gamma a + 1)$ and longitudinal magnetic fields are multiplied by the coefficient $(a + 1)$. These are well known facts. They apply equally to the relativistic Stern-Gerlach force. Expressing matters in the lab frame K, the equations of motion for the momenta $p_{x,y}$, for the coupling to the spin, are

$$\frac{dp_x}{dt} = -\frac{\partial(\boldsymbol{\Omega} \cdot \boldsymbol{s})}{\partial x} = -\left(\frac{\partial \boldsymbol{\Omega}}{\partial x}\right) \cdot \boldsymbol{s} \,, \qquad \frac{dp_y}{dt} = -\left(\frac{\partial \boldsymbol{\Omega}}{\partial y}\right) \cdot \boldsymbol{s} \,. \tag{1}$$

- Consider motion through a quadrupole and treat only p_x below. Then to a sufficient approximation for this note, with $-e\boldsymbol{B}/(p_s c) = K_1(x\boldsymbol{e}_y + y\boldsymbol{e}_x)$,

$$\boldsymbol{\Omega} = -\frac{e}{mc}\frac{\gamma a + 1}{\gamma}\boldsymbol{B} = (\gamma a + 1) v_s K_1 (x\boldsymbol{e}_y + y\boldsymbol{e}_x) \,. \tag{2}$$

Then

$$\frac{dp_x}{dt} = -\left(\frac{\partial \boldsymbol{\Omega}}{\partial x}\right) \cdot \boldsymbol{s} = -(\gamma a + 1) v_s K_1 s_y \,. \tag{3}$$

[1] Or Thomas-BMT or Thomas-Frenkel-(anyone else you like)-BMT equation.

Say the quadrupole length is L_q. The time to transit the quadrupole is $\Delta t = L_q/v_s$. Then, approximately, the Stern-Gerlach momentum kick upon traversing the quadrupole is

$$\Delta p_x^{SG} \simeq \frac{dp_x}{dt}\,\Delta t \simeq -(\gamma a+1)L_q K_1 s_y\,. \tag{4}$$

The angular deflection is

$$\Delta\theta_x^{SG} = \frac{\Delta p_x^{SG}}{p_s} \simeq -\frac{\gamma a+1}{m\gamma v_s}L_q K_1 s_y\,. \tag{5}$$

- Talman's expression is

$$(\Delta\theta_x^{SG})_{\rm Talman} = -\frac{\mu_x^*}{e\beta_s}\,q_x\,. \tag{6}$$

Here $q_x = 1/f$ is the inverse focal length. (I have changed Talman's "v" to β_s, see below.)

- Let us compare the two expressions. There are some issues about normal and skew quadrupoles, but that is not essential here. Talman states that "μ_x^* and μ_y^* differ from the Bohr magneton *(not magnetron)* μ_B only by $\sin\theta$ and $\cos\theta$ factors respectively." Here $\theta = \pi/4$ and is not important. Then, approximately, $L_q K_1 \simeq 1/f = q_x$. Also $\mu_x^* \simeq (e/mc)s_x$, up to factors of $1/\sqrt{2}$. Then I obtain

$$\Delta p_x^{SG} \simeq -\frac{\gamma a+1}{\gamma}\,\frac{q_x s_y}{m v_s} \tag{7}$$

Talman's expression is

$$(\Delta\theta_x^{SG})_{\rm Talman} = -\frac{1}{mc\beta_s}\,q_x s_x = -\frac{q_x s_x}{m v_s}\,. \tag{8}$$

See my comment above about β_s. I ignore global minus signs and the fact that my expression has s_y and Talman's contains s_x; this may be because I treated a normal quadrupole and Talman treated a skew quadrupole. Both "s_x" and "s_y" are simply spin components, of $O(\hbar/2)$. For the CEBAF injection line, $E \simeq 123$ MeV so $\gamma \simeq 240$ (and $\gamma a \simeq 0.25$, which can be ignored). Hence overall

$$\Delta p_x^{SG} \simeq \frac{\mathbf{1}}{\boldsymbol{\gamma}}\frac{q_x}{m v_s}\,\frac{\hbar}{2}\,, \qquad (\Delta\theta_x^{SG})_{\rm Talman} \simeq \frac{q_x}{m v_s}\,\frac{\hbar}{2}\,. \tag{9}$$

- *The angular deflection due to the Stern-Gerlach force is a factor* $1/\gamma$ *relative to that derived by Talman.* For a 123 MeV beam line this is about a factor of 240.

- The effects of the Lorentz boosts have not been calculated correctly. For relativistic particles, the Stern-Gerlach force is not proportional to the magnetic dipole moment. There may be other errors.