Nuclear Inst. and Methods in Physics Research, A 875 (2017) 165–176

Contents lists available at ScienceDirect

Nuclear Inst. and Methods in Physics Research, A

journal homepage: www.elsevier.com/locate/nima

RF resonant beam polarimetry: Analysis using quantized operators

S.R. Mane [a,*], W.W. MacKay [b]

[a] Convergent Computing Inc., P. O. Box 561, Shoreham, NY 11786, USA
[b] 25 Rhododendron Circle, Asheville, NC 28805, USA

A R T I C L E I N F O

Keywords:
Polarized beams
Storage rings
Resonant depolarizer
Cavity QED
Jaynes–Cummings model
Rabi oscillations

A B S T R A C T

The concept of so-called 'rf resonant beam polarimetry' has been proposed as a potentially fast, accurate and nondestructive technique for measuring the spin polarization of stored polarized beams. The published analyses have employed a semiclassical treatment for the cavity rf fields and also the particle spin. We revisit the problem, using quantized operators for the cavity rf field, and also treat the particle spin as a quantum operator. With suitable approximations, the quantum model can be solved exactly, yielding so-called 'vacuum Rabi oscillations.' Using our solution of the quantum model, we are able to offer more precise quantitative estimates for the energy and number of photons emitted into the cavity per unit time. Our treatment employing quantized operators yields significantly different conclusions from the semiclassical analysis.

1. Introduction

The concept of 'rf resonant beam polarimetry' was proposed by Derbenev [1,2] as a potentially fast, accurate and nondestructive technique for measuring the spin polarization of stored hadron beams. (In principle, the technique can also be applied for e^+e^- beams, but fast and accurate nondestructive polarimetry based on QED processes such as laser Compton backscattering is available for such beams.) Briefly, Derbenev's proposal [1] is to place an rf cavity in the ring, where the frequency of the resonant cavity mode would equal the spin precession frequency (plus an integer multiple of the beam revolution frequency). The cavity parameters are chosen so that the cavity has no modes resonant with the orbital motion, i.e. the beam revolution frequency and the betatron and synchrotron oscillation frequencies. The cavity is also 'passive' in the sense that it is initially empty. The circulating spins interact resonantly with the cavity and deposit electromagnetic energy into the cavity, essentially an inverse of the more usual situation of a spin flipper or rf depolarizer, where it is the electromagnetic fields in the cavity which drive the spin precessions. The electromagnetic energy in the cavity is proportional to the voltage in the cavity and is in turn proportional to the beam polarization. See [1, Secs. 2–4] for details. Additional technical details were elaborated in [2].

Derbenev's analysis in [1,2] was semiclassical (also called quasi-classical below). The electromagnetic field in the cavity was treated classically. The spins of the particles were treated as classical vectors, of length $\frac{1}{2}\hbar$, executing classical rotations. Planck's constant, or rather $\hbar$, appears only as a formal parameter, to specify the magnitude of a particle spin vector. In practice, however, in view of the small magnitude of the estimated electromagnetic energy deposited in the cavity, we decided to revisit the problem and reformulate the analysis using quantized operators for the electromagnetic field in the cavity and also to treat the particle spins as quantum operators. We treat only the case of spin $\frac{1}{2}$ and the vector polarization. (Derbenev also treated only the vector polarization.) We found that there is a natural analogy of the spin–cavity interaction with the so-called Jaynes–Cummings model [3]. The latter model treats the interaction of a nonrelativistic two-level atom in a resonant cavity, and was developed in the 1960s in connection with the then new subject of laser physics. Instead of a two-level atom executing atomic transitions between a ground state and excited state, we have a spin flips between 'up' and 'down' spin states. In particular, using the approximations made by Derbenev in [1], the quantum model can in fact be solved exactly, i.e. the eigenstates and eigenvalues of the spin–cavity Hamiltonian can be calculated exactly. The result is a direct analogy with the phenomenon of so-called 'vacuum Rabi oscillations'. Our solution for the vacuum Rabi oscillations will be displayed below. Derbenev [1] employed the example of a cavity with a TM_{110} resonant mode to perform numerical estimates. We employ the same TM_{110} cavity to obtain numerical estimates using our solution for the vacuum Rabi oscillations. We find that the estimated number of photons emitted into the cavity, on the timescale of the polarimetry measurement, is much less than unity. This is the principal finding of our paper, and is in sharp contrast to the results of Derbenev's analysis. Basically,

* Corresponding author.
E-mail addresses: srmane001@gmail.com (S.R. Mane), wwmackay@charter.net (W.W. MacKay).

http://dx.doi.org/10.1016/j.nima.2017.09.030
Received 3 November 2016; Received in revised form 1 September 2017; Accepted 14 September 2017
Available online 20 September 2017
0168-9002/

a semiclassical analysis yields that the electromagnetic energy in the cavity is a continuous function of time (starting from zero, because by definition the cavity is initially empty). However, the semiclassical result is given by an average over many quantum interactions. In reality, the interaction of the spins and the cavity consists of discrete spin flips, which yield photon emissions into the cavity. We conclude that the use of a resonant cavity as a polarimeter is significantly more complicated than Derbenev's analysis indicates [1,2].

Our purpose in this paper is to analyze the model treated by Derbenev [1,2], using quantum mechanics. As stated above, we show that the use of quantum mechanics is essential, i.e. the semiclassical analysis employed in [1,2] is not adequate to treat the problem. For this reason, in this paper we treat the same model employed by Derbenev and we make the same approximations as those in [1,2]. We recognize that in practice those conditions are difficult to satisfy in a real physical system. It is possible that there exist other storage ring designs, where resonant polarimetry would be easier to accomplish. We leave such issues to future work.

We also treat a number of related issues in this paper. For example, spin flippers and rf depolarizers are equivalent to driven resonant cavities, resonant with the spin precession frequency, and it is well known that a semiclassical model works well to treat both the electromagnetic fields in the cavity and for the spin precessions. This is well known in the context of the Froissart–Stora formula [4]. However, for spin flippers and rf depolarizers, the resonant cavity contains a macroscopic electromagnetic field and is *not* initially empty. We show how our quantum model yields the semiclassical results for spin flippers and rf depolarizers, when a suitable limit is taken. For this purpose, it is essential to recognize that the electromagnetic field in the cavity should be expressed using the so-called 'coherent states' of quantum optics [5,6]. We shall define coherent states and demonstrate the importance of their role in our analysis below.

Another property of the spin–cavity interaction, well known to workers in the field, is that the cavity electromagnetic field can kick the particle orbit. This drives a coherent betatron oscillation around the ring circumference, which causes the particle orbit to pass off-center through the ring quadrupoles. This modifies the effective spin resonance strength. See [7] for a review of the subject. The matter was mentioned by Derbenev in [2] and noted as a complication in the analysis. We also study the matter below, and point out that because the cavity electromagnetic field is quantized, the change to the particle orbits must also be treated quantum mechanically, yielding a further complication to the analysis.

We also note that Schottky signals can, in principle, also cause energy to be deposited in the cavity, due to the synchrotron oscillations and the bunched nature of the stored beam [8,9]. However, a simplified analysis, using parameters for RHIC, indicates that if the fractional spin tune is close to $\frac{1}{2}$ and the synchrotron tune is very small, as is typically the case in hadron rings, the Schottky power at the cavity resonant frequency is negligible.

The structure of our paper is as follows. In Section 2 we present our basic notation and definitions. Many of these follow Derbenev [1], to make contact with his work. In Section 3 we review the solution of Derbenev's quasiclassical model in [1], while in Section 4 we introduce the quantum model. In view of its importance to our paper, we place the solution of the quantum model in a separate section, in Section 5, where we derive the Rabi oscillations and also show how the quasiclassical limit may be obtained from the quantum model. Section 6 presents some numerical estimates, using a model of a cavity with a TM_{110} resonant mode (the same as that employed by Derbenev [1]). In Section 7 we analyze the interaction of the cavity with the beam current and in Section 8 we briefly discuss the driving of the coherent orbital oscillations. Section 9 presents an analysis of synchrotron oscillations and Schottky signals. Section 10 concludes.

2. Basic notation and definitions

2.1. General

We refer the reader to the texts by Conte and MacKay [10] and Dragt [11] for a general introduction to charged particle motion in accelerators, while [12] is a good reference for classical Hamiltonian dynamics. For a review of spin dynamics in accelerators, we refer the reader to [13,14]. We treat a particle of mass m and charge e, with velocity $\boldsymbol{v} = \boldsymbol{\beta} c$ and Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$. We denote the canonical particle coordinate and momentum by $\boldsymbol{r}$ and $\boldsymbol{p}$, respectively. We assume the orbital motion is integrable and denote the orbital action–angle variables by $(\boldsymbol{I}, \boldsymbol{\phi})$. As in [1], we treat particles of spin $\frac{1}{2}$ only, hence the polarization density matrix is fully specified by the vector polarization. We denote the semiclassical spin vector by $\boldsymbol{S}$, with amplitude $\hbar/2$, and the quantum spin operator by $\frac{1}{2}\hbar\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a vector of Pauli matrices. The magnetic moment anomaly is denoted by $G = (g-2)/2$ and the spin tune on an orbit by ν (and on the reference orbit by ν_0). In general we append a subscript '0' to denote the values of variables on the reference orbit. The coordinate axes are $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{s}}, \hat{\boldsymbol{y}})$ which are respectively radial (outwards), longitudinal and vertical. The positive sense of circulation is counterclockwise around the ring. We shall also use $\hat{\boldsymbol{z}}$ instead of $\hat{\boldsymbol{s}}$ to denote the longitudinal direction, e.g. σ_z for the rms bunch length. The independent variable is the time t and a dot denotes a time derivative. We shall also employ θ as the generalized azimuth around the ring, where $\theta = 2\pi s/L$, where $s = v_0 t$ is the arc length along the reference orbit and L is the ring circumference. The beam revolution angular frequency is $\omega_0 = 2\pi v_0/L$. The Hamiltonian can be expressed as

$$H = H_{\rm orb} + H_{\rm spin} + H_{\rm cav} + H_{\rm cav-orb} + H_{\rm cav-spin}\,. \tag{2.1}$$

Here $H_{\rm orb}$ describes the orbital motion of the particle, $H_{\rm spin}$ describes the spin motion in the prescribed accelerator (or 'external') guide fields, $H_{\rm cav}$ describes the cavity EM fields and the interaction terms are $H_{\rm cav-orb}$ and $H_{\rm cav-spin}$ for the interaction of the cavity fields with the particle orbit and spin, respectively. The interaction with free radiation fields (photon emission) is neglected. Following [1], we assume the cavity has no modes which are resonant with the orbital revolution frequency or the betatron and synchrotron oscillations. Note that, in general, the transverse magnetic fields in a resonant cavity will kick the orbit and drive coherent orbital oscillations around the ring. The resulting coherent orbital oscillations cause the spins to interact with the quadrupole, etc. magnetic fields around the ring, which modifies the spin resonance strength. This effect is not explicitly included in Eq. (2.1) and will be analyzed later. The accelerator guide fields are specified by a vector potential $\boldsymbol{A}_{\rm ext}$ and we neglect electrostatic fields. The orbital motion may be treated classically. To make contact with [1], we employ units such that the Lorentz force is given by $\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B})$. Hence the Hamiltonian for the orbital motion in the accelerator guide fields is

$$H_{\rm orb} = \left[\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}_{\rm ext}\right)^2 c^2 + m^2c^4\right]^{1/2}. \tag{2.2}$$

2.2. Poisson brackets, equations of motion and dynamical invariants

Note that in [1], the Poisson Brackets of a canonically conjugate (coordinate, momentum) pair (Q, P) is defined to be $\{P, Q\} = 1$ and Hamilton's equation for a dynamical variable D is $\dot{D} = \{H, D\}$. Derbenev's definitions follow the convention in [15]. We comment on this below. In the quantum theory, it is usual to write $[Q_q, P_q] = i\hbar$ (appending a subscript 'q' to denote quantum operators). Hence Derbenev's convention implies $[Q_q, P_q] = -i\hbar\{Q, P\}$. For the components of the spin operator, however, Derbenev writes (see [1] for details of definitions) $\{S_n, \hat{S}\} = i\hat{S}$ and $\{\hat{S}, \hat{S}^*\} = 2iS_n$. In the quantum theory, the corresponding commutator relations are $[S_{n,q}, \hat{S}_q] = -\hbar \hat{S}_q$ and $[\hat{S}_q, \hat{S}^*_q] = -2S_{n,q}$. This implies that for the spin components in [1],

$$[S_{n,q}, \hat{S}_q] = i\hbar\{S_n, \hat{S}\}, \qquad [\hat{S}_q, \hat{S}^*_q] = i\hbar\{\hat{S}, \hat{S}^*\}. \tag{2.3}$$

Hence there is a relative inconsistency of a minus sign in [1], for the relation between the quantum commutators and classical Poisson Brackets, when treating orbital and spin variables.

We employ a consistent treatment and define the Poisson Bracket as $\{Q, P\} = 1$. For all classical variables A_c and B_c and their quantum counterparts A_q and B_q, the relation between the quantum commutators and classical Poisson Brackets is defined to be $[A_q, B_q] = i\hbar\{A_c, B_c\}$. Hamilton's equation of motion for a classical function F_c is given by

$$\frac{dF_c}{dt} = \{F_c, H_c\} + \frac{\partial F_c}{\partial t}. \tag{2.4}$$

Here H_c is the classical Hamiltonian. The Heisenberg equation of motion for the corresponding quantum operator F_q, with H_q as the quantum Hamiltonian, is

$$i\hbar\frac{dF_q}{dt} = [F_q, H_q] + i\hbar\frac{\partial F_q}{\partial t}. \tag{2.5}$$

A (classical) dynamical variable D is a function of the coordinates and momenta (including the field in the cavity) and the spin, but is explicitly independent of the time t. Hence we say Hamilton's equation for a dynamical variable is $\dot{D} = \{D, H_c\}$. A dynamical invariant I is a dynamical variable whose value does not change with time, i.e. $\dot{I} = 0$, which implies $\{I, H_c\} = 0$. A dynamical invariant quantum operator I_q commutes with the Hamiltonian $[I_q, H_q] = 0$.

Synchrotron oscillations create some complications in the above formalism, because the Hamiltonian depends explicitly on the time, so $\dot{H} = \partial H/\partial t \neq 0$. Hence the total energy of the system is not conserved. It is assumed in [1] that the Hamiltonian is explicitly independent of the time, and we shall do the same. We shall discuss synchrotron oscillations later in this paper.

2.3. Spin basis

The spin precession vector in terms of electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ is given by

$$\boldsymbol{\Omega} = -\frac{e}{mc}\left[\left(G + \frac{1}{\gamma}\right)\boldsymbol{B}_\perp + \frac{1+G}{\gamma}\boldsymbol{B}_\parallel + \left(G + \frac{1}{\gamma+1}\right)\boldsymbol{E}\times\frac{\boldsymbol{v}}{c}\right]. \tag{2.6}$$

Here $\boldsymbol{B}_\perp$ and $\boldsymbol{B}_\parallel$ are the transverse and longitudinal components of $\boldsymbol{B}$ with respect to the particle velocity $\boldsymbol{v}$. We initially treat the spin as a quasiclassical vector. Then $H_{\text{spin}} = \boldsymbol{\Omega}_{\text{ext}} \cdot \boldsymbol{S}$ and $H_{\text{cav-spin}} = \boldsymbol{\Omega}_{\text{cav}} \cdot \boldsymbol{S}$, where $\boldsymbol{\Omega}_{\text{ext}}$ and $\boldsymbol{\Omega}_{\text{cav}}$ are obtained by substituting the accelerator guide fields and cavity fields in Eq. (2.6), respectively. With an obvious notation, $\boldsymbol{E}_{\text{ext}} = -(1/c)\partial\boldsymbol{A}_{\text{ext}}/\partial t$ and $\boldsymbol{B}_{\text{ext}} = \boldsymbol{\nabla}\times\boldsymbol{A}_{\text{ext}}$. The spin precession equation is given by the Thomas-BMT (Bargmann, Michel and Telegdi) equation [16,17], which in our model takes the form

$$\frac{d\boldsymbol{S}}{dt} = (\boldsymbol{\Omega}_{\text{ext}} + \boldsymbol{\Omega}_{\text{cav}})\times\boldsymbol{S}. \tag{2.7}$$

We assume the spin motion in the ring to be stable, i.e. nonresonant with the orbital revolution frequency and the betatron and synchrotron oscillations. By design, the spin precession frequency is resonant with a cavity mode, with angular frequency ω_c. The term in $\boldsymbol{\Omega}_{\text{cav}}$ will be treated as a perturbation. First we treat the 'unperturbed' spin precession, i.e.

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{\Omega}_{\text{ext}}\times\boldsymbol{S}. \tag{2.8}$$

We sketch only a summary below; the full details are given in [14]. The spin precession on a given orbit can be specified as a linear combination of three vectors $(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n})$, which form a right-handed orthonormal basis of solutions of Eq. (2.8). The unit vector $\boldsymbol{n}$ is the quantization axis of the spin eigenstates on the orbit [18] (see also the review [14]). Define also the vector $\boldsymbol{\eta} = \boldsymbol{l} + i\boldsymbol{m}$. Note that $\boldsymbol{\eta}\cdot\boldsymbol{\eta} = \boldsymbol{\eta}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{\eta}\cdot\boldsymbol{\eta}^* = 2$. Any solution of Eq. (2.8) can be parameterized via

$$\boldsymbol{S} = S_n\boldsymbol{n} + \frac{1}{2}\left(S_-\boldsymbol{\eta} + S_+\boldsymbol{\eta}^*\right). \tag{2.9}$$

Here $S_n = \boldsymbol{S}\cdot\boldsymbol{n}$, $S_+ = \boldsymbol{S}\cdot\boldsymbol{\eta}$ and $S_- = \boldsymbol{S}\cdot\boldsymbol{\eta}^*$. The nonvanishing Poisson Brackets are

$$\{S_n, S_\pm\} = \mp iS_\pm, \qquad \{S_+, S_-\} = -2iS_n. \tag{2.10}$$

On the reference orbit, the basis vectors have the periodicity property $\boldsymbol{n}_0(\theta+2\pi) = \boldsymbol{n}_0(\theta)$ and $\boldsymbol{\eta}_0(\theta+2\pi) = e^{-i2\pi\nu_0}\boldsymbol{\eta}_0(\theta)$. The periodicity properties for off-axis orbits are listed in [14]. The resonance condition with the cavity mode is

$$\omega_c = \nu_0\omega_0 + k\omega_0. \tag{2.11}$$

Here k is an integer. It will be convenient below to define $\nu_c = \omega_c/\omega_0$.

- The resonance condition was actually stated as an approximate equality in [1, eq. (3)], i.e. $\omega_c \simeq \nu_0\omega_0 + k\omega_0$. However, in the relevant derivations in [1, eq. (3)], Derbenev assumes $\omega_c = \nu_0\omega_0 + k\omega_0$ exactly (see below). Hence we assume that Eq. (2.11) is satisfied exactly. However, one should recognize that in real physical systems, the resonance condition will only be satisfied to within a very small tune spread.
- It is assumed in [1] that the resonance condition in Eq. (2.11) is unique. However, for a ring equipped with full strength Siberian Snakes, the (fractional) spin tune is $\nu_0 = \frac{1}{2}$, and the conditions for the spin resonance and 'mirror resonance' coincide: $\nu_c = \nu_0 + k = -\nu_0 + k + 1$. Then the resonance condition is not unique and the analysis in [1], or in this paper, is not applicable. This fact should be borne in mind.

We now observe that Derbenev [1] makes a very strict assumption in his model. He assumes that it is a good approximation that $\boldsymbol{n}$ takes the *same* value on the off-axis orbits, i.e. $\boldsymbol{n} = \boldsymbol{n}_0$ throughout the region of phase space where the spins interact significantly with the cavity. However, Derbenev's model permits a spin tune spread, i.e. $\nu - \nu_0$ may be nonzero off axis. Quoting from [1]: "We neglected here the spin–orbit coupling effect on S_n and $S_\pm$, but have taken into account the spin tune spread $\Delta\nu$". Following [1], we write $\boldsymbol{\eta}_0 = \boldsymbol{e}\,e^{-i\nu_0\theta}$. Notice that $\boldsymbol{e}(\theta + 2\pi) = \boldsymbol{e}(\theta)$. This vector will be employed later, for the cavity–spin interaction $H_{\text{cav-spin}}$. Following [1], we sum over the particles, indexed by $j = 1, \ldots, N$. Then the Hamiltonian for the 'free spin precession' is

$$\begin{aligned} H_{\text{spin}} &= \omega_0\sum_j(\nu_j + k)S^j_n \\ &= \omega_c\sum_j S^j_n + \sum_j \epsilon_j\, S^j_n. \end{aligned} \tag{2.12}$$

- The parameter ϵ_j yields the spin tune spread. It is given by [1]

$$\epsilon_j = (\nu_j - \nu_c + k)\,\omega_0. \tag{2.13}$$

  The term in ϵ_j in Eq. (2.12) will be treated as part of the interaction Hamiltonian below.
- Note that unless Eq. (2.11) is an exact equality, ϵ_j does *not* average to zero. The average is

$$\langle\epsilon_j\rangle = \frac{1}{N}\sum_j\epsilon_j = (\nu_0 - \nu_c + k)\,\omega_0. \tag{2.14}$$

  Later in [1], Derbenev *does* make the approximation $\epsilon_j = 0$ for all the particles, which is only possible if Eq. (2.11) is an exact equality. For these reasons, we choose to state the resonance condition Eq. (2.11) as an exact equality from the outset.
- Of course, in deriving Eq. (2.14), we assumed $\langle\nu_j\rangle = \nu_0$. It is possible for the spin distribution to have a systematic spin tune shift. However, such an effect is typically very small, proportional to the beam emittances, and we neglect it. Furthermore,

the approximation $\epsilon_j = 0$, which is employed in the analysis in [1], is not possible if the spin distribution has a systematic spin tune shift.

2.4. Field in cavity

We now treat the 'free field' cavity Hamiltonian $H_{\rm cav}$. We continue to employ a classical model. The vector potential $\boldsymbol{A}$ in the cavity satisfies the wave equation

$$\Delta\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = 0\,. \tag{2.15}$$

We employ the Coulomb (or radiation) gauge div$\boldsymbol{A} = 0$. We decompose $\boldsymbol{A}$ into a sum over the cavity modes. It suffices here to treat only the resonant mode. We express the vector potential for the resonant mode in the form $\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}^c(\boldsymbol{r})Q(t)$, i.e. separation of space and time variables. The electric field is given by

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} = -\frac{1}{c}\boldsymbol{A}^c(\boldsymbol{r})\frac{dQ}{dt}\,. \tag{2.16}$$

This is parameterized in [1] as

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{1}{c}P(t)\boldsymbol{E}^c(\boldsymbol{r})\,. \tag{2.17}$$

Hence $\boldsymbol{E}^c(\boldsymbol{r}) = \boldsymbol{A}^c(\boldsymbol{r})$ and $P(t) = \dot{Q}$. The magnetic field is given by

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A} = Q(t)\boldsymbol{\nabla}\times\boldsymbol{A}^c(\boldsymbol{r})\,. \tag{2.18}$$

This is parameterized in [1] as

$$\boldsymbol{B}(\boldsymbol{r},t) = Q(t)\boldsymbol{\nabla}\times\boldsymbol{E}^c(\boldsymbol{r}) \equiv Q(t)\boldsymbol{B}^c(\boldsymbol{r})\,. \tag{2.19}$$

The above expressions are consistent with the following equations in [1]

$$c^2\Delta\boldsymbol{E}^c + \omega_c^2\boldsymbol{E}^c = 0\,, \qquad {\rm div}\,\boldsymbol{E}^c = 0\,. \tag{2.20}$$

The normalizations of the eigenmodes are, correcting some misprints in [1],

$$\int(\boldsymbol{E}^c)^2\,d^3\boldsymbol{r} = 4\pi c^2\,, \qquad \int(\boldsymbol{B}^c)^2\,d^3\boldsymbol{r} = 4\pi\omega_c^2\,. \tag{2.21}$$

Then the Hamiltonian for the resonant mode of the cavity field is [19]

$$\begin{aligned} H_{\rm cav} &= \frac{1}{8\pi}\int\left[(\boldsymbol{E})^2 + (\boldsymbol{B})^2\right]d^3\boldsymbol{r} \\ &= \frac{1}{8\pi}\int\left[\frac{(\boldsymbol{E}^c)^2P^2(t)}{c^2} + (\boldsymbol{B}^c)^2Q^2(t)\right]d^3\boldsymbol{r} \\ &= \frac{1}{2}(P^2 + \omega_c^2Q^2)\,. \end{aligned} \tag{2.22}$$

This agrees with the expression in [1, eq. (1)]. Hamilton's equations are $\dot{Q} = \{Q, H_{\rm cav}\} = P$ and $\dot{P} = \{P, H_{\rm cav}\} = -\omega_c^2Q$, which together yield $\ddot{Q} = -\omega_c^2Q$, which when substituted into Eq. (2.15) yields Eq. (2.20). Next, to prepare the formalism for the interaction of the cavity field with the particle spins, we follow [1] and introduce the complex field amplitudes a and a^* via

$$a = \frac{iP + \omega_cQ}{\sqrt{2\omega_c}}\,e^{i\omega_ct}\,. \tag{2.23}$$

Their Poisson Bracket is $\{a, a^*\} = -i$. The electric and magnetic fields of the resonant mode are given by

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\sqrt{2\omega_c}}{c}\frac{ae^{-i\omega_ct} - a^*e^{i\omega_ct}}{2i}\boldsymbol{E}^c(r)\,, \tag{2.24a}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{ae^{-i\omega_ct} + a^*e^{i\omega_ct}}{\sqrt{2\omega_c}}\boldsymbol{B}^c(r)\,. \tag{2.24b}$$

The electromagnetic energy in the resonant mode of the cavity is $\mathscr{E}_c = \omega_c\,|a|^2$.

2.5. Spin–cavity interaction

We now treat the interaction of the particle spins with the resonant cavity mode. For simplicity, Derbenev [1] assumed the resonant electric field is zero on axis, and may be neglected. (This will be justified below.) Then using Eq. (2.6), (2.9) and (2.24b) and treating only one spin

$$\begin{aligned} H_{\rm spin-cav} &= -\frac{eQ}{mc}\left[\left(G + \frac{1}{\gamma}\right)\boldsymbol{B}_\perp^c + \frac{1+G}{\gamma}\boldsymbol{B}_\parallel^c\right]\cdot\boldsymbol{S} \\ &= -\frac{e}{mc}\frac{ae^{-i\omega_ct} + a^*e^{i\omega_ct}}{\sqrt{2\omega_c}}\left[\left(G + \frac{1}{\gamma}\right)\boldsymbol{B}_\perp^c + \frac{1+G}{\gamma}\boldsymbol{B}_\parallel^c\right] \\ &\quad\cdot\left(S_n\boldsymbol{n} + \frac{S_-\boldsymbol{\eta} + S_+\boldsymbol{\eta}^*}{2}\right) \\ &= -\frac{e}{2mc\sqrt{2\omega_c}}\left[\left(G + \frac{1}{\gamma}\right)\boldsymbol{B}_\perp^c + \frac{1+G}{\gamma}\boldsymbol{B}_\parallel^c\right] \\ &\quad\cdot\left(a^*S_-\boldsymbol{e}e^{ik\theta} + aS_+\boldsymbol{e}^*e^{-ik\theta}\right) + \text{nonresonant}\,. \end{aligned} \tag{2.25}$$

We must average the above around the ring. (The averaging procedure is well known, in the expansion of a localized resonance driving term as a sum of Fourier harmonics around the ring circumference, see [14] for details.) We also sum over the spins. Following [1], we write, discarding the nonresonant terms,

$$H_{\rm spin-cav} = g_ka^*\sum_j S_-^j + g_k^*a\sum_j S_+^j\,. \tag{2.26}$$

The coupling strength is given by [1, eq. (4)]

$$g_k = -\frac{e}{2mc\sqrt{2\omega_c}}\left\langle\left[\left(G + \frac{1}{\gamma}\right)\boldsymbol{B}_\perp^c + \frac{1+G}{\gamma}\boldsymbol{B}_\parallel^c\right]\cdot\boldsymbol{e}e^{ik\theta}\right\rangle. \tag{2.27}$$

The average is around the ring circumference.

- To derive the above expression, it is essential that the spin basis vectors are the same on all the particle orbits, otherwise g_k depends also on the orbit, and must be placed inside the sum over j.
- Note also that we have so far neglected the contribution from the coherent orbital oscillations to the resonance strength.

3. Solution of quasiclassical model

3.1. Interaction Hamiltonian and equations of motion

We summarize the solution of the quasiclassical model. The interaction Hamiltonian is given by the spin tune spread term in Eq. (2.12) and the spin–cavity coupling in Eq. (2.26) and is denoted by $H_{\rm res}$ in [1]:

$$H_{\rm res} = \sum_j \epsilon_j\,S_n^j + g_ka^*\sum_j S_-^j + g_k^*a\sum_j S_+^j\,. \tag{3.1}$$

We observe that $H_{\rm res}$ is only a subset of all the interaction terms, because numerous nonresonant interactions have been neglected. This is a standard approximation in perturbation theory. In the absence of interactions, the values of a, $S_\pm^j$ and S_n^j would be stationary. The terms in $H_{\rm res}$ cause their values to vary with time. The resulting equations of motions are

$$\dot{a} = \{a, H_{\rm res}\} = -ig_k\sum_j S_-^j\,, \tag{3.2a}$$

$$\dot{S}_-^j = \{S_-^j, H_{\rm res}\} = -i\epsilon_j\,S_-^j + i2g_k^*aS_n^j\,, \tag{3.2b}$$

$$\dot{S}_n^j = \{S_n^j, H_{\rm res}\} = ig_ka^*S_-^j - ig_k^*aS_+^j\,. \tag{3.2c}$$

These equations differ in the signs of some terms relative to [1, Eqs. (5–6)] because of our definitions of the Poisson Brackets and Hamilton's equations.

3.2. Invariants

It is stated in [1] that the above system has the following general invariants:

- The Hamiltonian $H_{\rm res}$ itself. Recall that $H_{\rm res}$ is only invariant to the extent that the nonresonant interactions were neglected.
- The total beam spin value at $\Delta\nu_j = 0$. We shall explain this statement below.
- The combined adiabatic invariant [1, eq. (7)]

$$I_+ = |a|^2 + \sum_j S_n^j \,. \tag{3.3}$$

We prove that I_+ is invariant as follows:

$$\begin{aligned}\dot{I}_+ &= \dot{a}a^* + a\dot{a}^* + \sum_j \dot{S}_n^j \\ &= -ig_k a^* \sum_j S_-^j + ig_k^* a \sum_j S_+^j + \sum_j \left(ig_k a^* S_-^j - ig_k^* a S_+^j\right) \\ &= 0\,.\end{aligned} \tag{3.4}$$

Next we explain the concept of 'the total beam spin value at $\Delta\nu_j = 0$'. Here '$\Delta\nu_j = 0$' means zero spin tune spread, i.e. $\epsilon_j = 0$ for all $j = 1, \ldots, N$. As we noted above, *this is only possible if the spin resonance condition in* Eq. (2.11) *is an exact equality*. Let us sum over the spins and define $\mathcal{S}_n = \sum_j S_n^j$ and $\mathcal{S}_\pm = \sum_j S_\pm^j$. The (squared) total spin vector is $\mathcal{S}^2 = \mathcal{S}_n^2 + \mathcal{S}_-\mathcal{S}_+$. If $\epsilon_j = 0$ then the Hamiltonian in Eq. (3.1) is a function of only the 'total spin components'

$$H_{\rm res}(\epsilon_j = 0) = g_k a^* \mathcal{S}_- + g_k^* a \mathcal{S}_+ \,. \tag{3.5}$$

From standard angular momentum theory, the square of total spin vector commutes with all of its components, i.e. classically $\{\mathcal{S}^2, \mathcal{S}_n\} = 0$ and $\{\mathcal{S}^2, \mathcal{S}_\pm\} = 0$. Hence $\{\mathcal{S}^2, H_{\rm res}\} = 0$, i.e. the 'the total beam spin value at $\Delta\nu_j = 0$' is invariant. Note that this invariance would *not* be possible if $\epsilon_j \neq 0$. Essentially, the analysis in [1] assumes the timescale for the decoherence of the spins is longer than the timescale of the use of the cavity for polarimetry.

3.3. Macroscopic field: rf depolarizer

It is instructive to solve the equations of motion for the case of an rf depolarizer. In that case, the cavity rf field is macroscopic and we may set the value of a to a fixed (macroscopic) constant, say a_m. Then Eq. (3.2a) yields a negligible perturbation of $O(N\hbar)$ to the value of a_m. We solve Eqs. (3.2b) and (3.2c) for S_n^j and S_-^j. We neglect the spin tune spread term ϵ_j. For brevity, set $ig_k^* a = |a_m g_k| e^{i\delta}$. Then

$$\dot{S}_-^j = 2|a_m g_k| e^{i\delta} S_n^j \,, \tag{3.6a}$$

$$\dot{S}_n^j = -|a_m g_k| e^{-i\delta} S_-^j - |a_m g_k| e^{i\delta} S_+^j \,. \tag{3.6b}$$

The solution is (here ϕ_j is a constant)

$$S_n^j = \frac{1}{2}\hbar \cos(2|a_m g_k| t + \phi_j)\,, \tag{3.7a}$$

$$S_-^j = \frac{1}{2}\hbar \sin(2|a_m g_k| t + \phi_j)\, e^{i\delta}\,. \tag{3.7b}$$

This is the well established solution for a resonance driving term operating without decoherence mechanisms: the spins all rotate at the same frequency, around a spin rotation axis determined by the resonance driving term. The spin resonance strength is $2|a_m g_k|/\omega_0$. The distribution of phases ϕ_j determines the degree of the initial beam polarization.

- In practice, the purpose of an rf depolarizer is to place the spins on resonance, where they decohere and depolarize quickly.
- The purpose of the above analysis was to confirm that the quasiclassical model works well for a macroscopic cavity field. The effect of the spins on the cavity field, via Eq. (3.2a), is negligible. We shall reproduce the above behavior when we treat the quantum model and take the classical limit.

3.4. Energy in initially empty cavity

We now solve for the energy emitted by the spins into an initially empty cavity. Using information from [1, Sec. 2], the initial conditions are

$$a(0) = 0\,, \qquad \sum_j S_n^j = \frac{N\hbar}{2}\,\xi\cos\alpha\,, \qquad \sum_j S_-^j = \frac{N\hbar}{2}\,\xi\sin\alpha\, e^{i\varphi}\,. \tag{3.8}$$

Quoting from [1, Sec. 2]: "ξ is the degree of beam polarization, α and φ are polar and azimuthal angles of coherent spin declination from the periodic axis $\boldsymbol{n}$". The 'coherent spin declination' is the polarization vector. The 'periodic axis $\boldsymbol{n}$' is really $\boldsymbol{n}_0$, since the model assumes that $\boldsymbol{n}$ is the same on all the orbits. We solve Eqs. (3.2a)–(3.2c) for small t and where the spin tune spread ϵ_j is neglected. Then, because $a(0) = 0$, both $\dot{S}_-^j = \dot{S}_n^j = 0$ at $t = 0$. Hence we approximate that the values of S_-^j and S_n^j do not change significantly for small t. Then the solution of Eq. (3.2a) is

$$a(t) \simeq -ig_k \frac{N\hbar}{2}\,\xi\sin\alpha\, e^{i\varphi}\, t\,. \tag{3.9}$$

The electromagnetic energy in the resonant mode of the cavity is [1, eq. (8)]

$$\mathscr{E}_c = \omega_c |a|^2 \simeq \omega_c N^2 |g_k|^2 \frac{\hbar^2 t^2}{4}\,\xi^2 \sin^2\alpha\,. \tag{3.10}$$

- The above solution is only valid to the extent that S_-^j and S_n^j are approximately constant. This will clearly not be true for all t. We also know that the spins can emit a maximum energy of $N\hbar\omega_c$, where all N spins flip, hence $\mathscr{E}_c \le N\hbar\omega_c$. Hence the criterion 'small t' must be quantified. Derbenev offers an analysis of the matter in [1].
- An immediate objection to the solution in Eq. (3.10) (or [1, eq. (8)]) is that it vanishes for $\alpha = 0$, even for a fully polarized beam $\xi = 1$. However, $\xi = 1$ and $\alpha = 0$ means all the spins are initially in the 'up' spin state (quantized along $\boldsymbol{n}$). The rate of emission into the cavity should be *maximum* in this case. Hence we require a more careful treatment of the problem, using a quantum model.

The extremely small values of $\Delta\nu$ given by Derbenev in [1, Table 1] require momentum spreads much smaller than are achievable in the listed rings. For example, in RHIC without Siberian Snakes, $\Delta\nu = G\gamma\ \sigma_p/p \simeq 0.24$, using the values for $G\gamma$ and σ_p/p in Table 2, which means that the spins of a bunch fully polarized in the longitudinal direction would decohere in about 4 turns –much less than the 0.06 s listed by Derbenev in [1, Table 1]. Even with Snakes, a simulation (treating only linear orbital dynamics) shows that it would take only about 100 turns to depolarize the beam with initial polarization perpendicular to the stable spin direction $\boldsymbol{n}$. (With Snakes, even though the design particle has $\nu = 0.5$ exactly, the spin tunes of the other particles vary from this and the spin tune spread is nonzero.) See Fig. 1, where a set of 1000 protons with an initial helicity of +1 were tracked in a model RHIC lattice with full strength Siberian Snakes (but no spin rotators). The parameter list for RHIC was given in Table 2. The details of the simulation are as follows. The lattice employed was a model of the RHIC Blue Ring, using only dipoles and quadrupoles and no higher multipoles. There were no lattice imperfections. The values of β_* at the various interaction points (IP) are listed in Table 3. Additional simulations showed that the decoherence was due almost entirely to the momentum spread of the particles. A simulation where the initial 95% normalized emittance was increased to 20π mm-mrad yielded almost the same results as those shown in Fig. 1. However, for a simulation where the initial relative momentum spread was reduced to $\sigma_p/p = 2.5 \times 10^{-4}$, the decoherence time approximately doubled.

Table 1
Values of relevant parameters (fundamental constants). The values quoted for the particle mass, Compton wavelength and magnetic moment anomaly are for a proton.

Parameter	Value	Unit
c	299 792 458	m/s
mc^2	938.272	MeV
$\lambda_C = 2\pi\hbar/(mc)$	1.321×10^{-15}	m
G	1.792847	
α_e^{-1}	137.035999139	
X_{11}	3.8317	
$J_2(X_{11})$	0.4028	

Table 2
Values of parameters for RHIC used in the text.

Parameter	Value	Unit
L	3833.845	m
N	10^{11}	
$G\gamma$	477.500	
E	249.90	GeV
f_0	78196	Hz
f_s	26.5	Hz
h_{inj}	360	rf harmonic
V_{rf}	280	kV
γ_t	23.57	
η_{ph}	-1.786×10^{-3}	$1/\gamma^2 - 1/\gamma_t^2$
σ_z	1.7	m
$\pi\epsilon_{x,y}$ (normalized, rms)	1.67π	mm-mrad
$\pi\epsilon_{x,y}$ (normalized, 95%)	10π	mm-mrad
σ_p/p	5.0×10^{-4}	
Q_x	28.695	
Q_y	28.685	
$\beta_{x,y}$	50.0	m
$y_{\text{c.o.}}$	0.1	mm
r_c	0.2	m

Table 3
Values of β^* at interaction points (IR).

IR	β_x^* (m)	β_y^* (m)
IR6 (STAR)	0.65	0.66
IR8 (PHENIX)	0.64	0.67
IR10	7.54	7.73
IR12	7.53	7.73
IR2	5.04	5.12
IR4	7.46	7.67

4. Quantum model

4.1. Spin precession

Other than the spin, we shall employ a hat to distinguish quantum operators from classical variables. The spin basis vectors, spin tune and resonance condition are the same in the quantum model. The corresponding quantum Hamiltonian is

$$\hat{H}_{\text{spin}} = \frac{1}{2}\hbar\omega_0\nu_0 \sum_j \sigma_n^j + \frac{1}{2}\hbar \sum_j \epsilon_j\, \sigma_n^j\,. \tag{4.1}$$

Here $\sigma_n = \boldsymbol{\sigma} \cdot \boldsymbol{n}_0$, $\sigma_+ = \boldsymbol{\sigma} \cdot \boldsymbol{\eta}_0$ and $\sigma_- = \boldsymbol{\sigma} \cdot \boldsymbol{\eta}_0^*$. The nonvanishing commutators are

$$[\sigma_n, \sigma_\pm] = \pm\sigma_\pm\,, \qquad [\sigma_+, \sigma_-] = 2\sigma_n\,. \tag{4.2}$$

4.2. Cavity field

Following the usage in quantum optics, laser physics and cavity quantum electrodynamics (QED), etc., we shall refer to the excitations of the cavity modes as 'photons' [3,5,6]. The cavity field variables are promoted to quantum operators $\hat{Q}$ and $\hat{P}$, with the commutator $[Q,P] = i\hbar$. The annihilation operator $\hat{a}$ of the resonant cavity field is given by (see Eq. (2.23))

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega_c}}(iP + \omega_c Q)\, e^{i\omega_c t}\,. \tag{4.3}$$

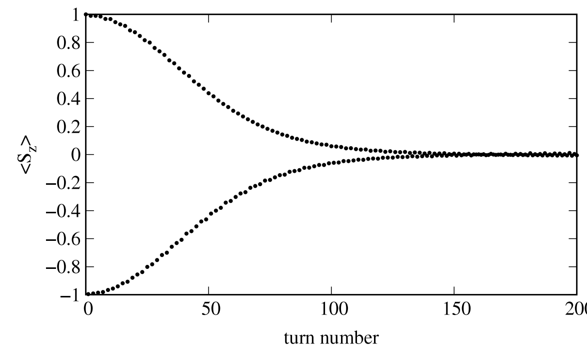

Fig. 1. Spin tracking simulation of 1000 protons with initial horizontal polarization and helicity of +1, in a model RHIC lattice with full strength Siberian Snakes. The (fractional) spin tune is $\frac{1}{2}$, which explains the two branches of the plot.

The creation operator is $\hat{a}^\dagger$ and $[\hat{a},\hat{a}^\dagger] = 1$. As with a and a^*, $\hat{a}$ and $\hat{a}^\dagger$ pertain only to the resonant cavity mode: there are corresponding operators for all the other cavity modes. The number operator of the photons in the resonant mode is $\hat{\mathcal{N}} = \hat{a}^\dagger\hat{a}$. The eigenvalues of $\hat{\mathcal{N}}$ are the non-negative integers $n = 0, 1, 2, \ldots$. The corresponding eigenstates $|n\rangle$ are known as Fock states. We shall require the following properties below:

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \qquad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle\,. \tag{4.4}$$

The quantum expectation value for electromagnetic energy in the resonant mode of the cavity is $\mathcal{E}_c = \hbar\omega_c\,\langle\hat{a}^\dagger\hat{a}\rangle$. Here the angle brackets denote an expectation value using the quantum state of the resonant cavity field. It is not an average over time, for example. Of course, the resonant cavity field will be created by interactions with all the particles, so the above average will implicitly include the particle orbits and spins. In a suitable limit, the above should equal the classical expression for the cavity energy. Let us analyze this limit in more detail.

The Fock states are eigenstates of the free-field cavity Hamiltonian $\hat{H}_{\text{cav}}$, and they form an orthonormal basis, but they do not have a classical interpretation. The expectation value of the cavity field in a Fock state $|n\rangle$ is zero. The electric and magnetic field operators are obtained from Eqs. (2.24) and (2.24b) with the substitution of $\sqrt{\hbar}\,\hat{a}$ and $\sqrt{\hbar}\,\hat{a}^\dagger$ for a and a^* respectively. Then, using $\langle n|\hat{a}|n\rangle = \langle n|\hat{a}^\dagger|n\rangle = 0$ yields

$$\langle n|\hat{\boldsymbol{E}}|n\rangle = \langle n|\hat{\boldsymbol{B}}|n\rangle = 0\,. \tag{4.5}$$

The quantum states which most closely resemble classical electromagnetic waves are called 'coherent states' (see [5,6]). The coherent states are eigenstates of the annihilation operator, i.e. $\hat{a}|\zeta\rangle = \zeta|\zeta\rangle$, where ζ is a complex number. To distinguish between Fock and coherent states, we employ Roman letters to denote a Fock state, e.g. $|n\rangle$, and Greek letters to denote a coherent state, e.g. $|\zeta\rangle$. Note that the ground Fock state $|0\rangle$ is also a coherent state. A (normalized) coherent state can be expressed as a sum of Fock states via

$$|\zeta\rangle = e^{-|\zeta|^2/2} \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{n!}}\,|n\rangle = e^{-|\zeta|^2/2} e^{\zeta\hat{a}^\dagger}\,|0\rangle\,. \tag{4.6}$$

Although the coherent states yield a complete set of states, they do not form a basis. The overlap between any two coherent states $|\zeta\rangle$ and $|\zeta'\rangle$ is nonzero: $|\langle\zeta'|\zeta\rangle|^2 = e^{-|\zeta-\zeta'|^2}$. The coherent states are said to be *overcomplete* [6]. If the cavity field is given by a coherent state $|\zeta\rangle$, the quantum expectation values of the cavity electric and magnetic fields are (using $\langle\zeta|\hat{a}|\zeta\rangle = \zeta$)

$$\langle\zeta|\boldsymbol{E}|\zeta\rangle = -\frac{\sqrt{2\hbar\omega_c}}{i2c}\left(\zeta e^{-i\omega_c t} - \zeta^* e^{i\omega_c t}\right) \boldsymbol{E}^c(r)\,, \tag{4.7a}$$

$$\langle\zeta|\boldsymbol{B}|\zeta\rangle = \sqrt{\frac{\hbar}{2\omega_c}}\left(\zeta e^{-i\omega_c t} + \zeta^* e^{i\omega_c t}\right) \boldsymbol{B}^c(r)\,. \tag{4.7b}$$

These expressions can be interpreted classically. If we set $a = \langle\zeta|\sqrt{\hbar}\,\hat{a}|\zeta\rangle = \sqrt{\hbar}\,\zeta$, we reproduce Eqs. (2.24a) and (2.24b). Note, however, that unlike a Fock state, a coherent state is *not* a state of definite photon number: a coherent state is not an eigenstate of the number operator $\hat{\mathcal{N}}$ (except for the ground state $|0\rangle$). The mean and variance of the number of photons are given by

$$\mu_\zeta = \langle\zeta|\hat{\mathcal{N}}|\zeta\rangle = |\zeta|^2\,, \qquad \sigma_\zeta^2 = \langle\zeta|\hat{\mathcal{N}}^2|\zeta\rangle - \mu_\zeta^2 = |\zeta|^2\,. \tag{4.8}$$

These results are characteristic of a Poisson distribution of the photon number, which can be derived from Eq. (4.6). For a macroscopic value $|\zeta| \gg 1$, the standard deviation is negligible compared to the mean photon number, and the expectation values in Eqs. (4.7a) and (4.7b) can be treated as deterministic classical fields. We shall make extensive use of coherent states below.

4.3. Interaction Hamiltonian

We make the same approximations used to derive Eq. (3.1). For brevity, define $\tilde{g}_k = \sqrt{\hbar}\,g_k$. The quantum interaction Hamiltonian is

$$\hat{H}_{\rm res} = \frac{1}{2}\hbar\Big(\sum_j \epsilon_j\,\sigma_n^j + \tilde{g}_k\hat{a}^\dagger\sum_j\sigma_-^j + \tilde{g}_k^*\hat{a}\sum_j\sigma_+^j\Big)\,. \tag{4.9}$$

As in the quasiclassical model, the quantum model has dynamical invariants. First, there is the Hamiltonian $\hat{H}_{\rm res}$ itself. Next, if $\epsilon_j = 0$ for all $j = 1,\dots,N$, then the (squared) total spin operator is invariant. Define the total spin components $\Sigma_n = \sum_j\sigma_n^j$ and $\Sigma_\pm = \sum_j\sigma_\pm^j$. Then define the squared total spin operator $\boldsymbol{\Sigma}^2 = \Sigma_n^2 + \frac{1}{2}(\Sigma_+\Sigma_- + \Sigma_-\Sigma_+)$. Note that if $\epsilon_j = 0$ then the interaction Hamiltonian reduces to

$$\hat{H}_{\rm res} = \frac{1}{2}\hbar\big(\tilde{g}_k\hat{a}^\dagger\Sigma_- + \tilde{g}_k^*\hat{a}\Sigma_+\big)\,. \tag{4.10}$$

Then $[\boldsymbol{\Sigma}^2, \Sigma_n] = [\boldsymbol{\Sigma}^2, \Sigma_\pm] = 0$, hence $[\boldsymbol{\Sigma}^2, \hat{H}_{\rm res}] = 0$. Third, the classical invariant I_+ has a quantum analog

$$\hat{I}_+ = \hbar\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\sum_j\sigma_n^j\,. \tag{4.11}$$

It is easily verified that $[\hat{I}_+, \hat{H}_{\rm res}] = 0$.

4.4. Quantum invariants

The notion of 'invariance' for a quantum operator must be understood carefully. In general, if an operator $\hat{J}$ is a quantum invariant, then $\hat{J}$ commutes with the Hamiltonian and the Heisenberg equation of motion yields $d\hat{J}/dt = 0$. However, what will measurements of the value of $\hat{J}$ yield? A classical dynamical invariant always has a definite value, which is specified by the initial conditions of the particle orbit and spin, and the cavity field. This is not always true for a quantum invariant. For example, consider the photon number operator $\hat{\mathcal{N}}$. This is a quantum invariant, if we treat only the cavity free field (no interactions). If the cavity field is given by a Fock state $|n\rangle$, then measurements of $\hat{\mathcal{N}}$ always yield the value n, because the quantum state $|n\rangle$ is an eigenstate of $\hat{\mathcal{N}}$. However, if the quantum state is a coherent state $|\zeta\rangle$, then $|\zeta\rangle$ is *not* an eigenstate of $\hat{\mathcal{N}}$ (except for $\zeta = 0$), and measurements of $\hat{\mathcal{N}}$ do not always yield a definite value.

In general, measurements of a quantum invariant operator $\hat{J}$ will yield a definite value only if the quantum system is in an eigenstate of $\hat{J}$. Otherwise, measurements of $\hat{J}$ will not always yield the same value. This is due to quantum uncertainty, and is not related to noise or interactions, etc. The strongest statement we can make is that, for any quantum state, the *expectation value* $\langle\hat{J}\rangle$ is constant in time.

- If $\epsilon_j = 0$, the squared total spin operator $\boldsymbol{\Sigma}^2$ will have a definite value only if the spins are in an eigenstate of the total spin. For $N = 2$, this means the singlet or triplet spin state, but not a linear combination of the two. In general, the operator $\boldsymbol{\Sigma}^2$ will not have a definite value for a partially polarized beam, although the (squared) classical total spin $\boldsymbol{\mathcal{S}}^2$ will do so.
- The quantum invariant $\hat{I}_+$ will have a definite value only if the cavity field is in a Fock state $|n\rangle$. However, Fock states do not have a classical limit, even for large n. To make contact with the classical invariant I_+, the cavity field must be described by a coherent state, but this is not an eigenstate of $\hat{I}_+$ and does not have a definite number of photons. The quantum invariant $\hat{I}_+$ will also have a definite value only if the spins are in an eigenstate of the spin operator $\sum_j S_n^j$. This will not be the case, for example, for a partially polarized beam.

5. Solution of quantum model

5.1. Rabi oscillations

We place the solution of the quantum model in a separate section. As a first step, we derive the so-called 'Rabi oscillations'. This is an important phenomenon, and the material below draws heavily from the Jaynes–Cummings model [3]. We begin with a model of one spin (and drop the index j on the particle). The eigenstates can be calculated exactly for this case. The one-spin model can be solved for nonresonant oscillations also, but we treat only the resonant case. The Hamiltonian is

$$\hat{H}_{\rm res} = \frac{1}{2}\hbar\big(\tilde{g}_k\hat{a}^\dagger\sigma_- + \tilde{g}_k^*\hat{a}\sigma_+\big)\,. \tag{5.1}$$

Suppose the initial cavity state is the Fock state $|n\rangle$. Denote the up and down spin states by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively. The above Hamiltonian couples the states $|\uparrow, n\rangle$ and $|\downarrow, n+1\rangle$, i.e. (spin up, n photons) and (spin down, $n+1$ photons). Note that these are joint (spin, field) quantum states. Our basis of states is $\binom{1}{0}$ which represents $|\uparrow, n\rangle$ and $\binom{0}{1}$ which represents $|\downarrow, n+1\rangle$. Denote the Hamiltonian in this basis by H_n. Then, using Eq. (4.4),

$$H_n = \hbar\begin{pmatrix} 0 & \tilde{g}_k^*\sqrt{n+1} \\ \tilde{g}_k\sqrt{n+1} & 0 \end{pmatrix}\,. \tag{5.2}$$

The eigenvalues of H_n are $\pm\hbar\Omega_{n+1}$, where $\Omega_n = |\tilde{g}_k|\sqrt{n}$. The corresponding eigenstates $|\psi_\pm\rangle$ are

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}\,|\tilde{g}_k|}\begin{pmatrix}\tilde{g}_k^* \\ \pm\tilde{g}_k\end{pmatrix}\,. \tag{5.3}$$

These are known as 'dressed states'. Suppose the initial state is $|\uparrow, n\rangle$ (spin up, n photons), then for $t \neq 0$ the state is

$$\begin{aligned}|\psi(t)\rangle &= \frac{\tilde{g}_k}{\sqrt{2}\,|\tilde{g}_k|}\Big[e^{-i\Omega_{n+1}t}|\psi_+\rangle + e^{i\Omega_{n+1}t}|\psi_-\rangle\Big] \\ &= \begin{pmatrix}\cos(\Omega_{n+1}t) \\ -\dfrac{i\tilde{g}_k^2}{|\tilde{g}_k|^2}\sin(\Omega_{n+1}t)\end{pmatrix}.\end{aligned} \tag{5.4}$$

The expectation value of the electromagnetic energy in the cavity resonant mode is

$$\begin{aligned}\mathcal{E}_c = \hbar\omega_c\,\langle\hat{a}^\dagger\hat{a}\rangle &= \hbar\omega_c\Big[n\cos^2(\Omega_{n+1}t) + (n+1)\sin^2(\Omega_{n+1}t)\Big] \\ &= \hbar\omega_c\Big[n + \sin^2(\Omega_{n+1}t)\Big]\,.\end{aligned} \tag{5.5}$$

The oscillation of the energy in the cavity is known as a 'Rabi oscillation' and the above results were derived in the Jaynes–Cummings model [3] for a two-level atom interacting with a resonant cavity.

5.2. Partially polarized beam

Suppose instead that the spin is initially down. The spin must absorb a photon to flip to the up state. Then the state space is $|\downarrow, n\rangle$ and $|\uparrow, n-1\rangle$.

It is easily derived that the expectation value of the electromagnetic energy in the cavity resonant mode is

$$\begin{aligned}\mathcal{E}_c &= \hbar\omega_c \left[n\cos^2(\Omega_n t) + (n-1)\sin^2(\Omega_n t) \right] \\ &= \hbar\omega_c \left[n - \sin^2(\Omega_n t) \right].\end{aligned} \tag{5.6}$$

Suppose the beam is partially polarized, with a projection $\cos\alpha$ along $\boldsymbol{n}_0$. This can be treated as a weighted statistical mixture of two beams, one polarized up and the other polarized down. The expectation value is

$$\begin{aligned}\mathcal{E}_c &= \hbar\omega_c \left[(n + \sin^2(\Omega_{n+1} t))\cos^2(\alpha/2) + (n - \sin^2(\Omega_n t))\sin^2(\alpha/2) \right] \\ &= \hbar\omega_c \left[n + \cos^2(\alpha/2)\sin^2(\Omega_{n+1} t) - \sin^2(\alpha/2)\sin^2(\Omega_n t) \right].\end{aligned} \tag{5.7}$$

The case of interest to us is $n = 0$, when the cavity is initially empty, and is known as a 'vacuum Rabi oscillation'. Then

$$\mathcal{E}_c = \hbar\omega_c \cos^2(\alpha/2)\sin^2(\Omega_1 t). \tag{5.8}$$

Experiments with two-level atoms have observed vacuum Rabi oscillations [20,21], in agreement with the theory [3].

5.3. Multiple particles

The above analysis is for only one spin. To generalize to $N > 1$ particles, we offer an approximate argument as follows. A maximum of N photons can be emitted, hence we say the number of emitted photons is $N/2$ 'on average'. Then we replace Ω_n by $|\tilde{g}_k|\sqrt{n + (N-1)/2}$, where we write $(N-1)/2$ so that the new definition matches the previous for $N = 1$. Then for an initially empty cavity, we obtain approximately

$$\begin{aligned}\mathcal{E}_c &\simeq N\hbar\omega_c \cos^2(\alpha/2)\sin^2(\sqrt{\hbar}|g_k|\sqrt{(N-1)/2}\,t) \\ &\simeq \omega_c N^2 |g_k|^2 \frac{\hbar^2 t^2}{4}(1 + \cos\alpha).\end{aligned} \tag{5.9}$$

In the last line we approximated $N \gg 1$ and also expanded for small t. The latter expression should be compared to the quasiclassical expression [1, eq. (8)]. Setting $\xi = 1$ in Eq. (3.10) yields

$$\mathscr{E}_c = \omega_c N^2 |g_k|^2 \frac{\hbar^2 t^2}{4}\sin^2\alpha. \tag{5.10}$$

- We remarked previously that the quasiclassical expression yields zero for $\alpha = 0$, i.e. when all the spins point up, which is erroneous. The solution using the quantum model has a factor $1 + \cos\alpha$, which is maximum for $\alpha = 0$, i.e. all spins up, and zero for $\alpha = \pi$, i.e. all spins down, which is correct.
- In the quasiclassical model, the value of the cavity field amplitude is independent of the spin direction, by definition. This is known to be a reasonable approximation for a macroscopic cavity field. However, for an initially *empty* cavity, the quantum model demonstrates that the state of the cavity field and the particle spin are strongly correlated. The quantum eigenstates, the so-called 'dressed states', are *joint* eigenstates of the cavity and the spin.
- As already noted, the quantum expression for the cavity energy is an *expectation* value. Measurements of the cavity energy will display statistical fluctuations due purely to quantum uncertainty, independent of noise in the cavity or fluctuations in the phase space distribution of the beam, etc. We shall comment on this below.

5.4. Macroscopic field: rf depolarizer

To make contact with the case of an rf depolarizer (macroscopic field), the quantum state should be a coherent state, say $|\zeta\rangle$ as in Eq. (4.6). This model can also be solved exactly for one spin. We are principally interested in the case where the initial spin state is $|\uparrow\rangle$. Then the 'up' and 'down' spin components of the quantum state are given by infinite sums over the Fock states as follows

$$\langle\uparrow|\psi(t)\rangle = e^{-|\zeta|^2/2}\sum_{n=0}^{\infty}\frac{\zeta^n}{\sqrt{n!}}\cos(\Omega_{n+1}t)\,|n\rangle, \tag{5.11a}$$

$$\langle\downarrow|\psi(t)\rangle = -\frac{i\tilde{g}_k^2}{|\tilde{g}_k|^2}\,e^{-|\zeta|^2/2}\sum_{n=0}^{\infty}\frac{\zeta^n}{\sqrt{n!}}\sin(\Omega_{n+1}t)\,|n+1\rangle. \tag{5.11b}$$

The above solution pertains to any coherent state. It was noted in [20], for example, that for a macroscopic coherent state, the above sum is strongly peaked at the average photon number $\bar{n} = |\zeta|^2 \gg 1$. Then the expectation of σ_n, for example, can be obtained using Eq. (5.4) with $\bar{n}$ and approximating $\bar{n} \gg 1$:

$$\langle\psi|\sigma_n|\psi\rangle \simeq \cos^2(\Omega_{\bar{n}+1}\,t) - \sin^2(\Omega_{\bar{n}+1}\,t) \simeq \cos(2|\tilde{g}_k|\sqrt{\bar{n}}\,t). \tag{5.12}$$

Next, set $\sqrt{\hbar}\,|\langle\zeta|\hat{a}|\zeta\rangle| = \sqrt{\hbar}\,|\zeta| = \sqrt{\hbar\bar{n}} = a_m$, were a_m is the same as in the quasiclassical model treated in Section 3.3. Then $|\tilde{g}_k|\sqrt{\bar{n}} = |g_k|\sqrt{\hbar\bar{n}} = |a_m g_k|$, hence

$$\langle\psi|\sigma_n|\psi\rangle \simeq \cos(2|a_m g_k|t). \tag{5.13}$$

This matches the solution for S_n^j in Eq. (3.7) (with the initial phase $\phi_j = 0$). The corresponding expression for S_-^j can also be derived from the quantum model.

- We estimate the average photon number in a macroscopic cavity field as follows. A table of data for numerous spin flippers is given in [22]. We may say that a typical peak integrated magnetic field for a spin flipper or rf depolarizer is 1 T-mm. From information in [23], the lengths of an rf solenoid and rf dipole spin flipper employed at the IUCF Cooler were about 50 cm and 40 cm, respectively. Hence a reasonable estimate for the cavity photon wavelength is $\lambda = 1$ m. It should be noted that the above values are merely approximate, but we are only making order of magnitude estimates. The momentum of one photon is $2\pi\hbar/\lambda$. We saw above that the resonance strength scales as $\sqrt{\bar{n}}$. Hence the estimated average photon number $\bar{n}$ is

$$\begin{aligned}\bar{n} &\simeq \left(\frac{eBL\lambda}{2\pi\hbar c}\right)^2 \simeq \left(\frac{10^6/3.3356}{2\pi \times 6.582\times 10^{-16}\times 2.997\times 10^8}\right)^2 \\ &\simeq 5.85\times 10^{22}.\end{aligned} \tag{5.14}$$

- If the spin were initially down, we would substitute $\Omega_{\bar{n}}$ for $\Omega_{\bar{n}+1}$. However, for $\bar{n} \gg 1$ the two frequencies are almost equal. This justifies the assumption in the quasiclassical model that the cavity field is independent of the state of the particle spin. As we noted above, such an approximation is not valid if the cavity is initially empty.
- The above analysis was for a model with one spin. The number of spins in a particle bunch is $N = O(10^{11})$, hence $\bar{n} \gg N \gg 1$. Hence the amplitude of the cavity field or the precession frequency $|\tilde{g}_k|\sqrt{\bar{n} + (N-1)/2}$ is not significantly affected by the presence of multiple spins. This justifies the neglect of the effect of the spins on the cavity field, for practical designs of spin flippers or rf depolarizers and typical bunch intensities.

6. Estimates for resonant cavity energy

6.1. Coupling strength

We now derive expressions for the coupling strength $|g_k|$ and numerical estimates for the electromagnetic energy in the resonant mode of the cavity. To do so, and for contact with [1], we treat a cylindrical cavity of length d and radius r_c with a resonant TM_{110} mode. We employ cylindrical coordinates (r, ϑ, z) where $\hat{\boldsymbol{z}}$ is directed along the ring reference axis and ϑ should not be confused with the ring azimuth

ϑ. For definiteness, we assume the magnetic field points radially in the horizontal plane and $\boldsymbol{n}_0$ is vertical at the location of the cavity. The vector potential is $\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}^c(\boldsymbol{r})e^{i\omega_c t}$, where

$$\boldsymbol{A}^c(\boldsymbol{r}) = A_0 J_1\left(\frac{X_{11}}{r_c}\,r\right)\,\sin\vartheta\,\hat{\boldsymbol{z}}\,. \tag{6.1}$$

Here A_0 is a normalization constant, J_1 is a Bessel function and X_{11} denotes its first positive zero. Note that $\omega_c r_c/c = X_{11}$. Then from Section 2.4, $Q(t) = e^{i\omega_c t}$ and $\boldsymbol{E}^c(\boldsymbol{r}) = \boldsymbol{A}^c(\boldsymbol{r})$. The electric field vanishes on axis, hence its coupling with the spins is negligible, as assumed in [1] and the analysis above. The magnetic field components are given by

$$B_r^c = -\frac{A_0}{r}\,J_1\left(\frac{X_{11}}{r_c}\,r\right)\,\cos\vartheta\,, \tag{6.2a}$$

$$B_\vartheta^c = -A_0\,\frac{X_{11}}{r_c}\,J_1'\left(\frac{X_{11}}{r_c}\,r\right)\,\sin\vartheta\,. \tag{6.2b}$$

A prime denotes differentiation with respect to the argument. Using Eq. (2.21), the normalization is

$$A_0 = \sqrt{4\pi c^2}\left(\frac{2}{\pi r_c^2 d J_2^2(X_{11})}\right)^{1/2} = \frac{2c}{J_2(X_{11})r_c}\,\sqrt{\frac{2}{d}}\,. \tag{6.3}$$

The transverse beam size is very small ($\sigma_{x,y} \ll r_c$) so it is a good approximation to write

$$\frac{A_0}{r}\,J_1\left(\frac{X_{11}}{r_c}\,r\right) \simeq \frac{A_0 X_{11}}{2r_c}\,. \tag{6.4}$$

The magnetic field is uniform along $\hat{\boldsymbol{z}}$. Integrate from $-d/2$ to $d/2$ to obtain

$$\frac{1}{L}\int_{-d/2}^{d/2} e^{i\omega_c z/v_0}\,dz = \frac{2v_0}{\omega_c L}\,\sin\frac{\omega_c d}{2v_0}\,. \tag{6.5}$$

Then

$$\begin{aligned} |g_k| &\simeq \left(G+\frac{1}{\gamma}\right)\frac{e}{2mc}\,\frac{1}{\sqrt{2\omega_c}}\,\frac{A_0 X_{11}}{2r_c}\,\frac{2v_0}{\omega_c L}\left|\sin\frac{\omega_c d}{2v_0}\right| \\ &= \frac{1}{J_2(X_{11})}\left(G+\frac{1}{\gamma}\right)\frac{e}{mc}\,\frac{\beta_0}{X_{11}}\,\sqrt{\frac{\omega_c}{d}}\,\frac{1}{L}\left|\sin\frac{\omega_c d}{2c\beta_0}\right|. \end{aligned} \tag{6.6}$$

The expression given in [1, eq. (9)] is (note that Derbenev writes '0.41' but in fact $J_2(X_{11}) \simeq 0.403$)

$$|g_k| = \frac{1}{J_2(X_{11})}\left(G+\frac{1}{\gamma}\right)\frac{e}{mc}\,\sqrt{\frac{2\pi\omega_c}{d}}\,\frac{1}{L}\left|\sin\frac{\omega_c d}{2c\beta_0}\right|. \tag{6.7}$$

This differs from our expression by a factor of $\sqrt{2\pi} \simeq 2.5$ in place of $\beta_0/X_{11} \simeq 0.26$ (with $\beta_0 \simeq 1$), about a factor of 9.6 larger. A private communication to Derbenev to clarify the issue has not yielded a response. We shall employ Eq. (6.6) below.

6.2. Numerical estimate of oscillation period

For an initially empty cavity, the period of the Rabi oscillations is approximately

$$T = \frac{\pi}{|\tilde{g}_k|\sqrt{N/2}}\,. \tag{6.8}$$

Recall $|\tilde{g}_k| = \sqrt{\hbar}\,|g_k|$. In our units, the electromagnetic fine structure constant is $\alpha_e = e^2/(\hbar c)$, hence

$$\frac{\sqrt{\hbar}\,e}{mc} = \frac{\lambda_C}{2\pi}\,\sqrt{c\alpha_e}\,. \tag{6.9}$$

Here $\lambda_C = 2\pi\hbar/(mc)$ is the Compton wavelength of the particle. We employ Eq. (6.6) and set $\beta_0 = 1$ and $\omega_c = X_{11}c/r_c$. Then

$$|\tilde{g}_k| = \frac{1}{J_2(X_{11})}\left(G+\frac{1}{\gamma}\right)\sqrt{\frac{\alpha_e}{X_{11}r_c d}}\,\frac{c\lambda_C}{2\pi L}\left|\sin\frac{X_{11}d}{2r_c}\right|. \tag{6.10}$$

Following [1, Sec. 5], we set $d = \pi\beta_0 c/\omega_c \simeq \pi r_c/X_{11}$. Using the parameter values in Tables 1 and 2 yields

$$|\tilde{g}_k| \simeq 1.8\times 10^{-11}\quad\text{Hz}\,. \tag{6.11}$$

The oscillation period is approximately

$$T \simeq 7.9\times 10^5\quad\text{s}\,. \tag{6.12}$$

6.3. Numerical estimate of photons in cavity

We treat the most favorable case of a fully polarized beam with all the spins initially up. We also write $\mathcal{E}_c = \hbar\omega_c\mathcal{N}_c$, so $\mathcal{N}_c$ is the estimated number of photons in the cavity resonant mode. We employ Eq. (5.9) to derive

$$\mathcal{N}_c = N^2|\tilde{g}_k|^2\,\frac{t^2}{2}\,. \tag{6.13}$$

We employ the parameter values in Tables 1 and 2 and the value of $|\tilde{g}_k|$ derived above. We use $t = 0.06$ s, from [1, Table 1]. Then

$$\mathcal{N}_c \simeq 5.7\times 10^{-3}\,. \tag{6.14}$$

The estimated number of photons in the cavity is much less than unity. For a partially polarized beam, the above estimate should be multiplied by a factor $(1+\cos\alpha)/2$.

- This is the main finding of our analysis. The above result is reminiscent of the photoelectric effect: although the quasiclassical model yields a smooth function of time, in practice the photons are emitted into the cavity in discrete integer units. This demonstrates that an analysis using the quantum model is essential.
- The quasiclassical expression is obtained by averaging over many photon emissions. However, the average must be understood carefully. It is not the case that multiple photons are emitted into the cavity in one measurement period. On any one run, at most one spin–flip photon will be emitted into the resonant mode of the cavity, during the time of the measurement. As opposed to the analysis in [1], which assumes a small but continuously varying cavity voltage, there will be either zero photons or one photon in the cavity, on any one run. With the small rates, this leads to prohibitively long data collection for any reasonable accuracy.

We close this section with a few additional remarks. First, the condition $d = \pi r_c/X_{11}$, with $r_c = 20$ cm, yields a cavity length of $d \simeq 16.4$ cm. The bunch length in RHIC is longer than this. (The numerical parameter values for RHIC are given in Table 2.) In fact, the value $r_c = 20$ cm violates the condition $\ell_b = \sigma_z \ll r_c$ stated in [1], that the bunch length should be much shorter than the cavity radius. (Possibly the condition was intended to state $\ell_b \ll d$, but r_c and d are of comparable magnitude in [1].[1]) Using data from Table 2, for $G\gamma \simeq 477.5$ the spin precession frequency is $G\gamma f_0 \simeq 37$ MHz. If we equate the cavity resonant frequency to this value, the resulting cavity radius is $r_c \simeq 4.9$ m and $d = \pi r_c/X_{11} \simeq 4.0$ m. These parameter values satisfy the conditions $\sigma_{x,y} \ll r_c$ and $\sigma_z \ll d$, but the cavity radius is large. Using the more 'reasonable' values $r_c = 20$ cm and $d = 1$ m yields

$$|\tilde{g}_k| \simeq 1.1\times 10^{-12}\quad\text{Hz}\,, \tag{6.15a}$$

$$T \simeq 1.3\times 10^7\quad\text{s}\,, \tag{6.15b}$$

$$\mathcal{N}_c \simeq 2.2\times 10^{-5}\,. \tag{6.15c}$$

The overall conclusion is the same: the expectation value of the number of spin–flip photons emitted into the cavity during a measurement period is much less than unity.

7. Interaction of cavity with beam current

7.1. General

The beam circulates around the ring at the revolution frequency and as such it acts as a periodic driving term to deposit energy in the cavity.

[1] The motivating idea is that the entire bunch should be contained inside the cavity, and in the transverse plane the bunch should be close to the cavity axis.

This is a nonresonant interaction, because by hypothesis the cavity has no modes which resonate with integer multiples of the revolution frequency. Our goal here is to estimate the nonresonant background field in the cavity originating from the circulating beam current. Since the current is macroscopic, the calculation may be formulated classically. Let $\boldsymbol{j}$ be the circulating current density and $\boldsymbol{A}_{\rm cav}$ the vector potential of the cavity field. We treat $\boldsymbol{j}$ as a prescribed function, i.e. we neglect the no back reaction of the cavity on the beam current. From Maxwell's equations, the cavity vector potential satisfies the equation [19]

$$\nabla^2 \boldsymbol{A}_{\rm cav} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}_{\rm cav} = -\frac{4\pi}{c}\,\boldsymbol{j}(\boldsymbol{r},t)\,. \tag{7.1}$$

We employ the Coulomb gauge, which is standard practice. The current density is multiplied by a periodic δ-function to indicate that the interaction only occurs in a localized region (the cavity). This yields a "comb" of Fourier harmonics at integer multiples of the revolution frequency. Hence the current density actually has the form

$$\boldsymbol{j}(\boldsymbol{r},t)\,\delta_p(\theta-\theta_{\rm cav}) = \boldsymbol{j}(\boldsymbol{r})\sum_{j=-\infty}^{\infty} e^{ij(\omega_0 t-\theta_{\rm cav})}\,. \tag{7.2}$$

The index j here should not be confused with the index $j = 1,\dots,N$ used previously to index the particles in the beam.

7.2. Free cavity field

Without the beam current, the cavity vector potential satisfies the free-space wave equation

$$\nabla^2 \boldsymbol{A}_{\rm cav} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}_{\rm cav} = 0\,. \tag{7.3}$$

We decompose $\boldsymbol{A}_{\rm cav}$ into a sum of modes $\boldsymbol{\mathcal{A}}_k(\boldsymbol{r})e^{i\omega_k t}$, indexed by a label k. (This label k should not be confused with that in Eq. (2.11) for the spin resonance condition.) Then $\boldsymbol{\mathcal{A}}_k(\boldsymbol{r})$ satisfies the eigenvalue equation

$$\nabla^2\boldsymbol{\mathcal{A}}_k + \frac{\omega_k^2}{c^2}\boldsymbol{\mathcal{A}}_k = 0\,. \tag{7.4}$$

This equation is solved using the boundary conditions of the cavity, to derive all the modes. From that we deduce the mode frequency ω_k (which is the eigenvalue). We assume the $\boldsymbol{\mathcal{A}}_k$ are normalized

$$\int_{\rm cav}\boldsymbol{\mathcal{A}}_{k_1}(\boldsymbol{r})\,\boldsymbol{\mathcal{A}}_{k_2}(\boldsymbol{r})\,d^3\boldsymbol{r} = \delta_{k_1k_2}\,. \tag{7.5}$$

The modes are also a complete set of functions in the cavity: they form a normalized basis.

7.3. Interaction

With the beam current, the cavity vector potential satisfies the inhomogeneous wave equation

$$\nabla^2 \boldsymbol{A}_{\rm cav} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}_{\rm cav} = -\frac{4\pi}{c}\,\boldsymbol{j}(\boldsymbol{r})\sum_{j=-\infty}^{\infty} e^{i(j\omega_0 t-\theta_{\rm cav})}\,. \tag{7.6}$$

We express the time dependence of $\boldsymbol{A}_{\rm cav}$ via a sum over j

$$\boldsymbol{A}_{\rm cav}(\boldsymbol{r},t) = \sum_{j=-\infty}^{\infty}\boldsymbol{A}_{\rm cav,\,j}(\boldsymbol{r})\,e^{i(j\omega_0 t-\theta_{\rm cav})}\,. \tag{7.7}$$

Then

$$\nabla^2\boldsymbol{A}_{\rm cav,\,j} + \frac{(j\omega_0)^2}{c^2}\boldsymbol{A}_{\rm cav,\,j} = -\frac{4\pi}{c}\,\boldsymbol{j}(\boldsymbol{r})\,. \tag{7.8}$$

To solve this we employ the mode expansion for the vector potential. We expand $\boldsymbol{j}(\boldsymbol{r})$ in a sum over the cavity modes

$$\boldsymbol{j}(\boldsymbol{r}) = \sum_k d_k\boldsymbol{\mathcal{A}}_k(\boldsymbol{r})\,. \tag{7.9}$$

Here the d_k are constant coefficients (independent of space and time). Since the modes form a normalized basis, we invert to obtain

$$d_k = \int_{\rm cav}\boldsymbol{j}(\boldsymbol{r})\cdot\boldsymbol{\mathcal{A}}_k(\boldsymbol{r})\,d^3\boldsymbol{r}\,. \tag{7.10}$$

We also expand $\boldsymbol{A}_{\rm cav,\,j}$ in a sum over the cavity modes

$$\boldsymbol{A}_{\rm cav,\,j}(\boldsymbol{r}) = \sum_k f_{jk}\,\boldsymbol{\mathcal{A}}_k(\boldsymbol{r})\,. \tag{7.11}$$

Here f_{jk} is also a constant (independent of space and time). Substituting the above expression into the equation for $\boldsymbol{A}_{\rm cav,\,j}$ and equating coefficients yields the solution

$$f_{jk} = \frac{4\pi c\,d_k}{\omega_k^2-(j\omega_0)^2}\,. \tag{7.12}$$

By design, the cavity has no modes which are resonant with integer multiples of the beam circulation frequency, hence the denominator does not vanish. We sum over j and k to obtain the nonresonant solution for the cavity vector potential

$$\boldsymbol{A}_{\rm cav}(\boldsymbol{r},t) = 4\pi c\sum_{j,k}\frac{d_k}{\omega_k^2-(j\omega_0)^2}\,\boldsymbol{\mathcal{A}}_k(\boldsymbol{r})\,e^{ij\omega_0 t}\,. \tag{7.13}$$

This has a macroscopic magnitude, but it is nonresonant with the spin precession frequency.

7.4. Numerical estimate

We employ a cylindrical cavity and treat the TM_{110} mode. We employ the expressions in Section 6 for the vector potential. For the TM_{110} mode,

$$\mathcal{A}_{110} = A_{110}J_1\left(\frac{X_{11}}{a}r\right)\sin\vartheta\,. \tag{7.14}$$

The normalization coefficient is

$$A_{110} = \left(\frac{2}{\pi r_c^2 d J_2^2(X_{11})}\right)^{1/2}. \tag{7.15}$$

We next need to evaluate the overlap integral

$$d_k = \int_{\rm cav}\boldsymbol{j}\cdot\boldsymbol{\mathcal{A}}_{110}\,d^3\boldsymbol{r}\,. \tag{7.16}$$

To evaluate this integral we must make some assumptions about the beam current density. For simplicity, suppose the beam has a Gaussian charge density with standard deviations $\sigma_{x,y,z}$ horizontally, vertically and longitudinally. We assume the bunch is oriented parallel to the cavity axis and has a circular transverse profile so $\sigma_x = \sigma_y$. Using values for RHIC from Tables 1 and 2 we estimate

$$\sigma_{x,y} = \sqrt{\epsilon_x\beta_x/(\beta\gamma)} \simeq \sqrt{1.67\times10^{-6}\times50/266} \simeq 0.6\ \text{mm}, \tag{7.17}$$

which is much smaller than the cavity radius r_c. The bunch contains N particles and moves at speed v_0. Then

$$\boldsymbol{j}(r,\vartheta,z) = Nev_0\,\frac{e^{-x^2/(2\sigma_x^2)}e^{-y^2/(2\sigma_y^2)}e^{-z^2/(2\sigma_z^2)}}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\,\hat{\boldsymbol{z}}\,. \tag{7.18}$$

If the bunch were centered on the cavity axis, the overlap integral with the cavity mode would vanish. Let us postulate a closed orbit distortion, so the bunch axis is displaced vertically from the cavity axis by $y_{\rm c.o.}$. Since $y_{\rm c.o.} \ll r_c$ and $b \ll r_c$, it is a good approximation that $J_1(X_{11}r/r_c) \simeq X_{11}r/(2r_c)$ within the domain of integration. Then

$$J_1(X_{11}r/r_c)\sin\vartheta \simeq \frac{X_{11}}{2r_c}\,r\sin\vartheta = \frac{X_{11}}{2r_c}\,y\,. \tag{7.19}$$

Then the value of the overlap integral is, assuming the beam size is small compared to the cavity dimensions,

$$\begin{aligned} d_k &= \int_{\rm cav}\boldsymbol{j}\cdot\boldsymbol{\mathcal{A}}_{110}\,dx\,dy\,dz \\ &= A_{110}\,\frac{Nev_0}{2\pi\sigma_x\sigma_y}\int\frac{X_{11}}{2r_c}\,ye^{-x^2/(2\sigma_x^2)}e^{-(y-y_{\rm c.o.})^2/(2\sigma_y^2)}\,dx\,dy \\ &= A_{110}\,\frac{X_{11}y_{\rm c.o.}}{2r_c}\,Nev_0\,. \end{aligned} \tag{7.20}$$

To estimate the number of photons, we should derive the effective integrated peak magnetic field $|e\boldsymbol{B}_{110}L_{\rm cav}|$, but since $|\boldsymbol{B}_{110}| \propto |\boldsymbol{A}_{110}|$, we can work with the vector potential. Using Eq. (7.13) and integrating over the cavity for the TM_{110} mode, we obtain

$$\left| \int_{\mathrm{cav}} \boldsymbol{A}_{110}^2 \, d^3\boldsymbol{r} \right|^{1/2} = 4\pi c \frac{|d_k|}{|\omega_c^2 - (j\omega_0)^2|} \,. \tag{7.21}$$

We select the value of j such that $\omega_c - j\omega_0 \simeq \omega_0/2$. Then $\omega_c^2 - (j\omega_0)^2 \simeq \omega_c\omega_0$. Then (approximating $v_0 \simeq c$)

$$\begin{aligned} \frac{e}{c}\,|\boldsymbol{A}_{110}| &= \frac{4\pi e}{\sqrt{\pi r_c^2 d}} \, \frac{|d_k|}{|\omega_c^2 - (j\omega_0)^2|} \\ &= \frac{2Ne^2c}{\omega_c\omega_0} \, \frac{\sqrt{2}\,X_{11}}{J_2(X_{11}) r_c^2 d} \, \frac{y_{\mathrm{c.o.}}}{r_c} \\ &= \frac{\hbar\omega_c}{c} \, \frac{N\alpha_e}{\pi} \, \frac{\sqrt{2}}{X_{11}J_2(X_{11})} \, \frac{L}{d} \, \frac{y_{\mathrm{c.o.}}}{r_c} \,. \end{aligned} \tag{7.22}$$

The peak integrated magnetic field is approximately $|e\boldsymbol{B}_{110}L_{\mathrm{cav}}| = |e\boldsymbol{A}_{110}d\omega_c/c| = |\pi e\boldsymbol{A}_{110}|$. As with the quantum model of an rf depolarizer, we equate this to $\hbar\omega_c\sqrt{N_{\mathrm{ph}}}$, where N_{ph} is the number of photons. Then

$$N_{\mathrm{ph}} = \left(\frac{\sqrt{2}\,N\alpha_e}{X_{11}J_2(X_{11})} \, \frac{L}{d} \, \frac{y_{\mathrm{c.o.}}}{r_c} \right)^2 . \tag{7.23}$$

The relevant parameters values are listed in Tables 1 and 2, e.g. we estimate $y_{\mathrm{c.o.}} \simeq 0.1$ mm. Previously we followed [1] and set $d = \pi\beta_0 c/\omega_c$, but for simplicity we set $d = 1$ m here. Then

$$N_{\mathrm{ph}} \simeq 1.6 \times 10^{18} \,. \tag{7.24}$$

This is a large number, but recall these photons are nonresonant. Hence a cavity resonator with a high Q is required, to filter out these photons. It is estimated in [1] that for a superconducting cavity, a value of $Q \simeq 2 \times 10^{10}$ is possible.

8. Coherent orbital oscillations

It is well known that a transverse magnetic field in a spin flipper or rf depolarizer not only rotates the spin but also kicks the particle orbit. This drives a coherent orbital oscillation around the ring circumference, which causes the orbit to pass off-axis through the ring quadrupoles, etc. This yields an additional contribution to the spin resonance strength. A review of the relevant formalism is given in [7]. The contribution of the coherent orbital oscillations to the spin resonance strength was noted by Derbenev in [2], who remarked that this phenomenon would complicate the analysis (the analyzing power of the polarimeter).

We discuss the coherent orbital oscillations briefly below, using a quantum model. Since the cavity is initially empty, it is the spin flips which generate the cavity field to drive the forced orbital oscillations. First consider a model of a homogeneous vertical magnetic field. The solutions for the eigenstates and eigenvalues of the Dirac equation in a uniform vertical magnetic field are known. The paper by O'Connell [24] has the required answer, including an anomalous magnetic moment. (O'Connell [24] cites a paper by Ternov, Bagrov and Zhukovskii, which we have not been able to obtain.) O'Connell sets $\hbar = c = 1$. The Dirac equation with an anomalous magnetic moment μ is

$$i\,\frac{\partial\psi}{\partial t} = \left\{ \boldsymbol{\alpha}\cdot(\boldsymbol{p} - e\boldsymbol{A}) + \gamma_4 m + \mu\gamma_4\boldsymbol{\sigma}\cdot\boldsymbol{B} \right\} \psi \,. \tag{8.1}$$

Here $\boldsymbol{\sigma}$ is a 4×4 spin matrix. Define the Schwinger critical field $B_c = m^2c^3/(e\hbar)$. Then the energy levels of the positive and negative energy states are, restoring $\hbar$ and c explicitly,

$$E_n = \pm\left\{ p_z^2c^2 + \left[\sqrt{m^2c^4 + e\hbar c B(2n + \xi + 1)} + \xi\,\frac{Ge\hbar B}{2mc} \right]^2 \right\}^{1/2} . \tag{8.2}$$

Here $n = 0, 1, 2 \ldots$ is the principal quantum number, $\xi = \pm 1$ indexes spin up and down, and p_z is the momentum of the particle along the z axis.

- The essential point is that if a spin flips, the quantum state of the orbit changes by only about $\Delta n = \pm 1$. Hence our fundamental point in this section is to emphasize that the change to the orbit (the forced orbital oscillations) cannot be treated as a semiclassical function of time as in [2]. *The change to the orbit will be quantized.*
- Hence, in the analysis of the Rabi oscillations, it will be necessary to extend the quantum state of the system to be a joint (spin, orbit, cavity) state. A change of the spin state will be correlated with a change to both the orbital state and the cavity field. The interaction Hamiltonian must include terms from $H_{\mathrm{cav-orb}}$ in Eq. (2.1), which have been neglected up to now. The term in H_{spin} in Eq. (2.1) must be extended to include the spin–orbit coupling due to the coherent orbital oscillations.
- We do not propose to analyze the matter further in this paper. Our findings above indicate that the period of a vacuum Rabi oscillation, for an initially empty cavity, is very long, and the estimated number of spin–flip photons which will be emitted into the cavity, during the measurement time, is much less than unity. Furthermore, those findings were derived using a model with very strict assumptions on the spin tune spread, etc.

9. Synchrotron oscillations and Schottky signals

Up to now, our analysis has assumed the Hamiltonian is explicitly independent of the time. This assumption was also made by Derbenev [1]. In practice, the need for longitudinal focusing means that real rings contain rf cavities and the Hamiltonian depends explicitly on the time. Hence the total energy of a particle is not conserved. The dynamical invariant I_+ (see Eq. (3.3)) is also not conserved. In practice, the energy oscillations (i.e. synchrotron oscillations) are bounded so that the energy, for example, oscillates around an average value. Note also that in most hadron rings, the synchrotron tune is much less than unity. At RHIC, the synchrotron oscillation frequency is 26.5 Hz, yielding a synchrotron tune of $\nu_s \simeq 3.4 \times 10^{-4}$, so that the time variation is 'slow' in some sense.

However, we have seen that the analysis above, even with the use of a time independent Hamiltonian and numerous other strict approximations on the spin tune spread, etc., yield the result that the estimated number of photons emitted into the resonant rf cavity is very small, much less than unity, on the timescale of relevance for use for polarimetry. Note also that it is explicitly assumed in [1] that the cavity is not resonant with the synchrotron oscillation frequency. Hence, we conclude that the inclusion of synchrotron oscillations, although it will make a quantitative difference to the analysis, is unlikely to affect the above conclusions of our analysis significantly.

There is, however, one further detail which should be considered, viz. that of Schottky signals. It is possible that a circulating (bunched) stored beam can deposit energy into the resonant cavity via incoherent Schottky signals, and that the frequency of some of these photons can equal the spin resonant frequency. If so, the Schottky signals would be an additional complication to the use of the resonant cavity for polarimetry. For the theory of Schottky signals of stored beams, including bunched beams, we refer the reader to the excellent texts by Chattopadhyay [8] and Boussard [9]. See also [25] for the use of Schottky spectra for longitudinal impedance measurements at RHIC. We offer a brief analysis of the longitudinal Schottky signals below. The Schottky signals due to betatron motion are similar and do not add any essential material. First, we note that the longitudinal dynamics in real rings such as RHIC is complicated, and can exhibit features such as long-lived soliton modes [26]. It is not our purpose here to present a comprehensive theory of longitudinal dynamics or Schottky signals. We confine our analysis to the simplest case of small-amplitude incoherent synchrotron oscillations. We model the detector is a point function (localized detector). For simplicity we place the detector at the azimuth $\theta_{\text{det}} = 0$. The signal is detected at a harmonic number h_{Sch}. The revolution angular frequency of particle j is $\omega_j = \omega_0(1 + (\Delta f)_j/f_0)$. Let the phase slip factor be $\eta_{\text{ph}} = 1/\gamma^2 - 1/\gamma_t^2$, where γ_t is the transition gamma, so $(\Delta f)_j/f_0 = \eta_{\text{ph}}(\Delta p)_j/p_0$. The beam is bunched. It is sufficient for our purposes to treat only small amplitude synchrotron oscillations, hence we write

$$\frac{(\Delta p)_j}{p_0} = A_j \cos(\omega_s t + \phi_j)\,. \tag{9.1}$$

Here $A_j = \sqrt{2J_j}$ is the amplitude and $\langle J_j \rangle = \sigma_p^2/p_0^2$. The initial phase ϕ_j has a uniform random distribution in $[0, 2\pi)$. Then, using the Jacobi–Anger identity for Bessel functions, the Schottky signal from one particle is proportional to

$$\begin{aligned}\exp\left(ih_{\text{Sch}} \int^t \omega_j \, dt' \right) &= \exp\left(ih_{\text{Sch}}\omega_0 t + i\,\frac{h\omega_0\eta_{\text{ph}}A_j}{\omega_s}\,\sin(\omega_s t + \phi_j) \right) \\ &= e^{ih_{\text{Sch}}\omega_0 t} \sum_{m=-\infty}^{\infty} e^{im(\omega_s t+\phi_j)} J_m\left(\frac{h_{\text{Sch}}\eta_{\text{ph}}A_j}{\nu_s} \right).\end{aligned} \tag{9.2}$$

The use of m as a summation index should not be confused with the particle mass. The (longitudinal) Schottky signal consists of a sum of synchrotron sidebands around the parent line (which is at h times the beam circulation frequency), separated at intervals of the synchrotron oscillation frequency. The Schottky power in the mth sideband is, summing over all the particles and averaging over the phases ϕ_j,

$$P_{\text{Sch}}(h, m) = K_{\text{Sch}} \sum_{n=1}^{N} J_m^2\left(\frac{h\eta_{\text{ph}}A_j}{\nu_s} \right). \tag{9.3}$$

Here K_{Sch} is a constant. We evaluate the above as follows. Rigorous treatments can be found in [8] and [9] but for simplicity we assume a Gaussian beam phase space distribution. Then

$$\begin{aligned}P_{\text{Sch}}(h, m) &= N K_{\text{Sch}} \int_0^\infty J_m^2\left(\frac{h_{\text{Sch}}\eta_{\text{ph}}\sqrt{2J}}{\nu_s} \right) e^{-J/\langle J\rangle}\,\frac{dJ}{\langle J\rangle} \\ &= N\, e^{-\sigma^2} I_m(\sigma^2)\,.\end{aligned} \tag{9.4}$$

Here I_m is a modified Bessel function and

$$\sigma = \frac{h_{\text{Sch}}\eta_{\text{ph}}}{\nu_s}\,\frac{\sigma_p}{p_0}\,. \tag{9.5}$$

The ratio of the power in the mth sideband relative to the parent is $I_m(\sigma^2)/I_0(\sigma^2)$.

Let us make some numerical estimates using parameters for RHIC, using data from Table 2. We set the fractional spin tune to $\frac{1}{2}$, to be as far away as possible from an integer. Hence we set $G\gamma = 477.5$, i.e. a beam energy of 249.90 GeV, and take the cavity frequency as $f_{\text{cav}} = G\gamma f_0 = 37.3387$ MHz. The synchrotron oscillation frequency is $f_s = 26.5$ Hz. Then the value of the sideband m corresponding to a fractional tune of $\frac{1}{2}$ is given by

$$m = \frac{f_0}{2f_s} \simeq 1475\,. \tag{9.6}$$

Also $|\eta_{\text{ph}}| \simeq 0.001786$, while $h_{\text{Sch}} = 477$ and $\sigma_p/p_0 \simeq 5.0 \times 10^{-4}$. Then

$$\sigma \simeq \frac{477 \times 0.001786 \times 5.0 \times 10^{-4}}{3.39 \times 10^{-4}} \simeq 1.26\,. \tag{9.7}$$

For modified Bessel functions with arguments much smaller than the order, $I_m(z) \simeq (z/2)^m/m!$. Then the ratio of the power in the mth sideband relative to the parent is

$$\frac{I_m(\sigma^2)}{I_0(\sigma^2)} \simeq \frac{I_{1475}(1.58)}{I_0(1.58)} \simeq \frac{1}{1.728}\,\frac{0.79^{1475}}{1475!}\,. \tag{9.8}$$

This is a negligibly small ratio. Hence for the above set of parameter values, we may neglect the Schottky power at the spin resonant frequency of the cavity.

10. Conclusion

Our treatment employing quantized operators yields conclusions significantly different from Derbenev's semiclassical analysis [1,2]. Most important of all, a semiclassical treatment yields that the electromagnetic energy in the cavity is a smooth continuous function of time. However, the quantum model indicates that the energy will be emitted into the cavity discontinuously, in discrete quanta (photons), and the expected number of photons in the cavity is much less than unity. Furthermore, Derbenev made numerous approximations in his analysis. We employed the same approximations in our calculations using quantum operators. Derbenev's approximations are very strict, and are highly unlikely to be realized in practical or foreseeable designs of storage rings. If we consider realistic designs of storage rings, we conclude that the use of a resonant rf cavity for polarimetry is likely to be even more difficult than our findings above indicate.

Connections were established with important results in quantum optics [5,6], such as the Jaynes–Cummings model [3]. In particular, using the approximations made by Derbenev [1], the quantum model can be solved exactly, to derive the eigenstates and eigenvalues of the spin–cavity interaction. The resulting so-called 'vacuum Rabi oscillations' played a central role in our analysis. As a side issue, we demonstrated how the widely employed (and successful) semiclassical treatment of spin flippers and rf depolarizers may be obtained from the quantum model, via the use of coherent states to describe the cavity electromagnetic field.

References

[1] Ya.S. Derbenev, Nucl. Instrum. Methods A 336 (1993) 12–15.
[2] Ya. S. Derbenev, CERN SL/95-05 (AP) (1995) and Proc. XI Intl. Symp. High Energy Spin Physics, Bloomington, IN, (1994), AIP Conf. Proc. Vol. 343, 1995, pp. 264–272.
[3] E.T. Jaynes, F.W. Cummings, Proc. IEEE 51 (1963) 89–109.
[4] M. Froissart, R. Stora, Nucl. Instrum. Methods A 7 (1960) 297–305 (in French).
[5] R.J. Glauber, The quantum theory of optical coherence, Phys. Rev. 130 (1963) 2529–2539.
[6] M. Sargent III, M.O. Scully, W.E. Lamb Jr., Laser Phys. (1993) (Westview Press, Boulder, CO, USA).
[7] V.I. Ptitsyn, Yu.M. Shatunov, S.R. Mane, Nucl. Instrum. Methods A 608 (2009) 225–233.
[8] Swapan Chattopadhyay, Some Fundamental Aspects of Fluctuations and Coherence in Charged-Particle Beams in Storage Rings, 1984, CERN Report 84-11.
[9] D. Boussard, CERN 95-06, Vol. II, 1995, 749.
[10] M. Conte, W.W. MacKay, An Introduction to the Physics of Particle Accelerators, World Scientific Publishing Co. Pte. Ltd, Singapore, 2008.
[11] A.J. Dragt, Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics, University of Maryland, College Park, MD, USA, 2016 Available at http://www.physics.umd.edu/dsat/.
[12] H. Goldstein, C.P. Poole, J.L. Safko, Classical Mechanics, Addison-Wesley, Reading, 2001.
[13] S.R. Mane, Yu.M. Shatunov, K. Yokoya, J. Phys. G 31 (2005) R151–R209.
[14] S.R. Mane, Yu. M. Shatunov, K. Yokoya, Rep. Progr. Phys. 68 (2005) 1997–2265.
[15] L.D. Landau, E.M. Lifshitz, Mechanics, Pergamon Press, Oxford, 1960.
[16] L.H. Thomas, Phil. Mag. 3 (1927) 1–21.
[17] V. Bargmann, L. Michel, V.L. Telegdi, Phys. Rev. Lett. 2 (1959) 435–436.
[18] Ya.S. Derbenev, A.M. Kondratenko, Sov. Phys.–JETP 37 (1973) 968–973.
[19] J.D. Jackson, Classical Electrodynamics, third ed., Wiley, New York, 1998.
[20] M. Brune, et al., Phys. Rev. Lett. 76 (1996) 1800–1803.
[21] B.T.H. Varcoe, S. Brattke, M. Weidinger, H. Walther, Nature 403 (2000) 743–746.
[22] M.A. Leonova, et al., Phys. Rev. ST Accel. Beams 9 (2006) 051001.
[23] B.B. Blinov, RF Depolarizing Resonances in the Presence of a Full Siberian Snake and Full Snake Spin-Flipping, (Ph.D. thesis), Univ. Michigan, 2000 See also arXiv:physics/0304066 [physics.acc-ph].
[24] R.F. O'Connell, Phys. Rev. Lett. 21 (1968) 397–398.
[25] M. Blaskiewicz, et al., Longitudinal impedance measurement in RHIC, in: Proc. of EPAC 2002, Paris, 2002, pp. 1488–1490.
[26] M. Blaskiewicz, et al., Longitudinal solitons in RHIC, in: Proc. of PAC 2003, Portland, USA, 2003, pp. 3029–3031.