Energy-Angle Distribution of Thin Target Bremsstrahlung

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The differential bremsstrahlung cross section of Bethe and Heitler is integrated over scattered electron angles to obtain an expression for the distribution in energy and angle of the radiation from fast electrons in very thin targets. Screening is taken into account through the assumption of an atomic potential $(Ze/r)\exp(-r/a)$, and the calculation is restricted to high energies and to small to moderate angles. The result is the same as that of Sommerfeld for no screening, except that the argument of the logarithm now depends on angle as well as on energy. Integration of this expression over gamma-ray angle gives an analytic formula for the total intensity that is nowhere more than a few percent higher than the Bethe-Heitler result calculated numerically on the basis of the Thomas-Fermi potential.

THIS paper reports the result of the integration of the Bethe-Heitler differential bremsstrahlung cross section[1] over the angles of the scattered electron to obtain the distribution in energy and angle of the radiation from fast electrons in very thin targets. Targets thin enough for the present result to be directly applicable can be realized with linear accelerators, although probably not with betatrons or synchrotrons. This result may also prove to be of interest as a basis for further calculations on thick targets or on multiple traversals of thin targets. Calculations of this latter type that do not depend sensitively on the angle distribution of the primary bremsstrahlung have already been made. The effect of multiple scattering on the thick target angle distribution[2] has received satisfactory experimental verification,[3] as have also the effects of multiple scattering and radiation loss on the energy spectrum.[4]

In connection with the calculations reported in reference 2, an approximate expression for the energy distribution of the radiation in the forward direction was obtained but not published at the time. This formula[5] agrees with that obtained by setting $x=0$ (forward direction) in Eqs. (1) and (2) below, except that the constant C was estimated to be 191 instead of the more nearly correct value 111 given below. Except for very thin targets, however, this theoretical distribution would not be expected to give a precise account of the observed energy spectrum in the forward direction, because of multiple scattering and radiation loss in the target.[2,4]

In what follows, we use the notation of reference 1, according to which E_0 is the energy of the incident electron, E that of the scattered electron, $k=E_0-E$ that of the radiated quantum, $\mu=mc^2$ the rest energy of an electron, and θ_0 the angle between the quantum and the incident electron. The scattering atom, of atomic number Z, is represented by the potential $(Ze/r)\exp(-r/a)$, where a is chosen to be inversely proportional to the cube root of the atomic number, in general accord with the Thomas-Fermi model. In particular, we assume that $a=(C/137)(\hbar^2/me^2Z^{\frac{1}{3}})=C\hbar/mcZ^{\frac{1}{3}}$, where C is a dimensionless number of order 137, which is determined below by comparison with the numerical calculation of the energy spectrum when the screening is complete and the Thomas-Fermi potential is used. The atomic form factor that corresponds to the approximate potential assumed above is $F(q)=[1+(aq/\hbar c)^2]^{-1}$. The differential cross section[1] must then be multiplied by $[1-F(q)]^2$.

In the absence of screening ($F=0$), the integration over the angles associated with the scattered electron has been performed by Sommerfeld.[6] In that calculation, as well as in the present one, it is assumed that E_0, E, and k are all large in comparison with μ, and only leading terms are retained. Since most of the radiation comes off at angles $\theta_0 \lesssim \mu/E_0$, it turns out that neither calculation is valid for angles large in comparison with μ/E_0. The large angle distribution has been treated by Hough,[7] and makes only a higher order contribution to the integrated energy spectrum; since only large momentum transfers need be considered in that case, screening can be ignored, while the finite size of the nucleus, which can be neglected in the present calculation, must be taken into account.

In performing the integration over scattered electron angles, terms of order $(Z^{\frac{1}{3}}/C)^2$ are neglected in comparison with unity. Further, the result obtained is not accurate for angles $\theta_0 \lesssim Z^{\frac{1}{3}}/CE_0$ or $(\mu/E_0)^2$; however, these limiting angles are small enough to be of little physical interest, and the contribution to the integrated energy spectrum from this region is of higher order. It

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[5] G. D. Adams, Phys. Rev. **74**, 1707 (1948); Johns, Katz, Douglas, and Haslam, Phys. Rev. **80**, 1062 (1950).
[6] A. Sommerfeld, *Wellenmechanik* (Ungar, New York) or *Atombau und Spektrallinien* (Friedrich Vieweg & Sohn, Braunschweig, 1939), Vol. 2, p. 551.
[7] P. V. C. Hough, Phys. Rev. **74**, 80 (1948); G. Parzen, Phys. Rev. **81**, 808 (1951) shows that the errors inherent in the use of the Born approximation, on which the Bethe-Heitler formula is based, are much less important for small than for large angles.

is convenient to replace θ_0 by the reduced angle $x=E_0\theta_0/\mu$. The cross section that gives the energy-angle distribution is then

$$\sigma(k,x)dkdx=\frac{4Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{k}xdx\left\{\frac{16x^2E}{(x^2+1)^4E_0}-\frac{(E_0+E)^2}{(x^2+1)^2E_0^2}+\left[\frac{E_0^2+E^2}{(x^2+1)^2E_0^2}-\frac{4x^2E}{(x^2+1)^4E_0}\right]\ln M(x)\right\},\quad(1)$$

$$\frac{1}{M(x)}=\left(\frac{\mu k}{2E_0E}\right)^2+\left(\frac{Z^{1/3}}{C(x^2+1)}\right)^2.\quad(2)$$

This agrees with Sommerfeld's result[6] when Z is set equal to zero in the logarithm (no screening).

Equation (1) can be integrated over x to obtain the energy spectrum. It is sufficiently accurate to take the upper limit equal to ∞, in which case

$$dk\int_0^\infty\sigma(k,x)dx=\frac{2Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{k}\left\{\left(\frac{E_0^2+E^2}{E_0^2}-\frac{2E}{3E_0}\right)\times\left(\ln M(0)+1-\frac{2}{b}\tan^{-1}b\right)+\frac{E}{E_0}\left[\frac{2}{b^2}\ln(1+b^2)+\frac{4(2-b^2)}{3b^3}\tan^{-1}b-\frac{8}{3b^2}+\frac{2}{9}\right]\right\},\quad(3)$$

where $b=(2E_0EZ^{1/3}/C\mu k)$, and $M(0)$ is given by Eq. (2) with $x=0$. Equation (3) is also obtained if the form factor used here is substituted into Eq. (50) of Bethe's paper;[8] this provides a welcome check on Eqs. (1) and (2). The constant C can be evaluated by comparison with Eq. (54) of reference 8 for complete screening; this leads to[9]

$$C=183/e^{1/2}=111.$$

The validity of the approximate form factor used here can be estimated by comparing Eq. (3) with the numerical results, which are based on the Thomas-Fermi form factor presented in Fig. 1 and Table I of the paper by Bethe and Heitler.[10] With the choice of C above, the present results are correct for complete screening and for no screening. In between, Eq. (3) is larger than it should be by less than 2 percent for moderate values of Z, and is never more than 4 percent high in the worst case of large Z and energies such that the screening is incomplete (when the two factors on the right side of Eq. (2) have the same order of magnitude). It is useful to note that the square bracket term that multiplies E/E_0 in Eq. (3) equals 2/9 at $b=\infty$, decreases monotonically as b decreases, and can be neglected for $b<3$.

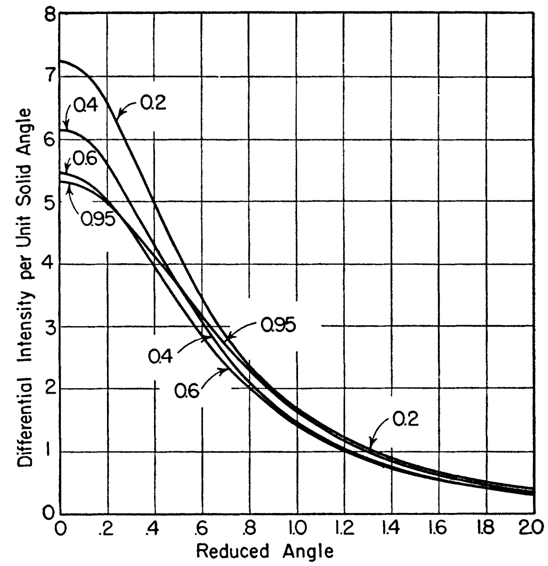

FIG. 1. Plots of the curly bracket in Eq. (1) against the reduced angle $x=E_0\theta_0/\mu$ for $Z=92$ and complete screening. The labels on the curves are values of k/E_0.

The curly bracket in Eq. (1), which is proportional to the differential gamma-ray *intensity* (not cross section) per unit solid angle, is plotted in Fig. 1 as a function of x for several values of k/E_0, for the particular case of complete screening and $Z=92$. Complete screening implies that $(2E_0E/\mu k)^2 \gg [C(x^2+1)/Z^{1/3}]^2$, so that in this case the plotted curves are independent of the value of E_0. For $x\leqq 2$, the ratio $[2E_0EZ^{1/3}/C(x^2+1)\mu k]^2$ is greater than or equal to 5 in the following cases: $k\leqq 0.95E_0$, $E_0\geqq 1300$ Mev; $k\leqq 0.9E_0$, $E_0\geqq 620$ Mev; $k\leqq 0.8E_0$, $E_0\geqq 270$ Mev; $k\leqq 0.6E_0$, $E_0\geqq 100$ Mev; $k\leqq 0.4E_0$, $E_0\geqq 46$ Mev; $k\leqq 0.2E_0$, $E_0\geqq 17$ Mev. In similar fashion, the left half of Fig. 1 ($x\leqq 1$) is a useful approximation for values of E_0 that exceed 40 percent of the limits quoted immediately above.

It is very likely that any reasonable choice of atomic form factor would alter only the dependence of $M(x)$ on x, given by Eq. (2), and not affect the rest of Eq. (1). However, a satisfactory proof of this point has not been devised; such a proof would probably make it possible to express $M(x)$ in terms of the atomic form factor.

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[8] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934).
[9] A small correction, neglected here, has been pointed out by J. H. Bartlett, Jr., Phys. Rev. **55**, 803 (1939).
[10] H. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).