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Ion Trapping and the magnetic mirror. Derivation follows from lecture notes by Dr. Peter T. Gallagher here: https://www.tcd.ie/Pl

1 Purpose

To derive criteria for two coaxial solenoids that are a relatively short distance apart to trap ions by creating a magnetic mirror. I'll start my derivation as much as possible from first principles and then apply them to the magnetic mirror model to determine whether or not it is realistically possible to trap ions in this manner. Constraints for mass (identity of the ion), parallel and perpendicular velocity (kinetic/thermal energy of the ion) are clearly necessary.

2 Derivation

Suppose we have a particle in a relatively uniform magnetic field that varies in magnitude with z, as is the case with two coaxial solenoids. In cylindrical coordinates we can write the components of the magnetic field:

$$\vec{B} = B_r\hat{r} + B_\theta\hat{\theta} + B_z\hat{z} \tag{1}$$

In the case of coaxial solenoids, $\vec{B}$ has azimuthal symmetry, thus $B_\theta = 0$. From $\nabla \cdot \vec{B} = 0$, we can obtain B_r:

$$\begin{aligned}
\nabla \cdot \vec{B} &= 0 \\
\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} &= 0 \\
\frac{\partial}{\partial r}(rB_r) &= -r\frac{\partial B_z}{\partial z} \\
rB_r &= -\int_0^r r'\frac{\partial B_z}{dz}dr'
\end{aligned}$$

If we assume that $\frac{\partial B_z}{\partial z}$ is known at $r = 0$ and does not vary significantly with r (a valid assumption in our case), then

$$\begin{aligned}
rB_r &\approx -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \\
B_r &= -\frac{r}{2}\left[\frac{\partial B_z}{\partial z}\right]_{r=0}
\end{aligned} \tag{2}$$

Now, in the absence of electric fields, the Lorentz force is

$$\begin{aligned}
\vec{F} &= q\vec{v} \times \vec{B} \\
&= q\left[(v_\theta B_z - v_z B_\theta)\hat{r} - (v_r B_z - v_z B_r)\hat{\theta} + (v_r B_\theta - v_\theta B_r)\hat{z}\right]
\end{aligned}$$

Since $B_\theta = 0$,

$$\vec{F} = q\left[v_\theta B_z\hat{r} + (v_z B_r - v_r B_z)\hat{\theta} - v_\theta B_r\hat{z}\right] \tag{3}$$

For the magnetic mirror, we are mainly concerned with F_z. Using (2), we have:

$$\begin{aligned}
F_z &= -qv_\theta B_r \\
&= \frac{qrv_\theta}{2}\frac{\partial B_z}{\partial z}
\end{aligned}$$

Averaging over one Larmor period, we have

$$F_{z,avg} = \pm\frac{qr_L v_\perp}{2}\frac{\partial B_z}{\partial z}$$

where $r_L = \frac{mv_\perp}{|q|\,B}$ is the Larmor radius. Plugging this in:

$$F_{z,avg} = -\frac{1}{2}\frac{mv_\perp^2}{B}\frac{\partial B_z}{\partial z}$$

But $\frac{1}{2}\frac{mv_\perp^2}{B} = \mu$, which is the magnetic moment of the particle. Thus,

$$F_{z,avg} = -\mu\frac{\partial B_z}{\partial z} \tag{4}$$

Equation (4) is the "mirror force". If we consider a line element ds along B, then we can extend $F_{z,avg}$ into 3D:

$$F_\parallel = -\mu\frac{dB_z}{ds} = -\mu\nabla_\parallel B$$

$F_\parallel$ is the mirror force parallel to B. Now,

$$\begin{aligned} F_\parallel = m\frac{dv_\parallel}{dt} &= -\mu\frac{dB}{ds} \\ mv_\parallel\frac{dv_\parallel}{dt} &= -\mu v_\parallel\frac{dB}{ds} \end{aligned}$$

Now, $\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = \frac{m}{2}\frac{d}{dt}\left(v_\parallel^2\right) = \frac{m}{2}\left(2v_\parallel\frac{dv_\parallel}{dt}\right) = mv_\parallel\frac{dv_\parallel}{dt}$, so

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu v_\parallel\frac{dB}{ds}$$

Since $v_\parallel = \frac{ds}{dt}$,

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{ds}{dt}\frac{dB}{ds} = -\mu\frac{dB}{dt}$$

Since the particle's energy is conserved in the absence of electric fields, $\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \mu B\right) = 0 \rightarrow \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\frac{d}{dt}(\mu B)$. Thus,

$$\begin{aligned} \mu\frac{dB}{dt} - \frac{d(\mu B)}{dt} &= 0 \\ \mu\frac{dB}{dt} - \left(B\frac{d\mu}{dt} + \mu\frac{dB}{dt}\right) &= 0 \\ B\frac{d\mu}{dt} &= 0 \end{aligned} \tag{5}$$

Since $B \neq 0$ by assumption, μ is a constant of the motion.

Let's apply this to the case of two coaxial solenoids. Suppose we have a charged particle between two coaxial solenoids that are relatively close together. Let the z-axis be the along the central axis of the solenoids. At two different locations z_1 and z_2, the strength of the magnetic field is B_1 and B_2 respectively, and the particle has transverse velocities $v_{\perp 1}$ and $v_{\perp 2}$ respectively. (Note that since there is azimuthal symmetry and we assume that the particle is close enough to the axis to feel the effects of the solenoidal magnetic fields, we only need to specify their z-component). Invoking the invariance of μ, we have

$$\begin{aligned} \mu_1 &= \mu_2 \\ \frac{mv_{\perp 1}^2}{2B_1} &= \frac{mv_{\perp 2}^2}{2B_2} \\ v_{\perp 1}^2 &= v_{\perp 2}^2\frac{B_1}{B_2} \end{aligned} \tag{6}$$

Also, since $\frac{dE}{dt} = 0$,

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}mv_{\|}^2 + \frac{1}{2}mv_{\perp}^2\right) &= 0\\ \frac{1}{2}mv_{\|}^2 + \frac{1}{2}mv_{\perp}^2 &= \varepsilon\\ v_{\|}^2 + v_{\perp}^2 &= \frac{2\varepsilon}{m}\end{aligned} \tag{7}$$

where ε is a constant. Suppose the particle moves from z_1 to z_2. If $B_1 < B_2$, then in order for equation (6) to be true, $v_{\perp 2} > v_{\perp 1}$, meaning that the particle's transverse velocity *increases*. By equation (7), this means that $v_{\|}$ *decreases* and the particle slows down. Solving (7) for $v_{\perp}^2$ and inserting it into (6), we have

$$\begin{aligned}\frac{2\varepsilon}{m} - v_{\|1}^2 &= \left(\frac{2\varepsilon}{m} - v_{\|2}^2\right)\frac{B_1}{B_2}\\ v_{\|1}^2 &= \frac{2\varepsilon}{m} - \left(\frac{2\varepsilon}{m} - v_{\|2}^2\right)\frac{B_1}{B_2}\\ v_{\|1}^2 &= \frac{2\varepsilon}{m}\left(1 - \frac{B_1}{B_2}\right) + v_{\|2}^2\frac{B_1}{B_2}\end{aligned}$$

The case when $v_{\|2} = 0$ corresponds to the case when the particle's velocity is solely *transverse* at z_2. In this case,

$$v_{\|1}^2 = \frac{2\varepsilon}{m}\left(1 - \frac{B_1}{B_2}\right) \tag{8}$$

where $B_1 < B_2$ by assumption so that the r.h.s. is positive.

Solving (8) for $\frac{B_1}{B_2}$ yields:

$$\begin{aligned}\frac{\frac{1}{2}mv_{\|1}^2}{\varepsilon} &= 1 - \frac{B_1}{B_2}\\ \frac{B_1}{B_2} &= 1 - \frac{\frac{1}{2}mv_{\|1}^2}{\varepsilon}\end{aligned} \tag{9}$$

Since ε is a constant,

$$\begin{aligned}1 - \frac{\frac{1}{2}mv_{\|1}^2}{\varepsilon} &= 1 - \frac{\frac{1}{2}mv_{\|1}^2}{\frac{1}{2}mv_{\|1}^2 + \frac{1}{2}mv_{\perp 1}^2}\\ &= \frac{\frac{1}{2}mv_{\|1}^2 + \frac{1}{2}mv_{\perp 1}^2}{\frac{1}{2}mv_{\|1}^2 + \frac{1}{2}mv_{\perp 1}^2} - \frac{\frac{1}{2}mv_{\|1}^2}{\frac{1}{2}mv_{\|1}^2 + \frac{1}{2}mv_{\perp 1}^2}\\ &= \frac{\frac{1}{2}mv_{\perp}^2}{\frac{1}{2}mv_{\|1}^2 + \frac{1}{2}mv_{\perp 1}^2}\\ &= \frac{\frac{1}{2}mv_{\perp 1}^2}{\varepsilon}\end{aligned}$$

and so

$$\boxed{\frac{B_1}{B_2} = \frac{\frac{1}{2}mv_{\perp 1}^2}{\varepsilon}} \tag{10}$$

$$\boxed{\varepsilon = \mu_1 B_2} \tag{11}$$

Eqs. (9) and (10) implies that the ratio $\dfrac{B_1}{B_2}$ determines the criteria for ions to be trapped in the mirror. Note that we tacitly assumed that the ions are collisionless. In reality, the velocities of the ions are constantly changing due to Coulomb collisions. The ions that are trapped can be freed through collisions and vice-versa. Thus, the magnetic mirror is unstable.