Josh Yoskowitz
Derivation of net force on ion between biased anode and magnetizing solenoid

1 Purpose

To derive an equation for the net force on an ion between the electric field from the biased anode and the magnetic field from the magnetizing solenoid. The resulting equation should be useful in determining the conditions for the ion to become trapped.

2 Derivation

We start with the experimental setup as shown below in Figure 1 below:

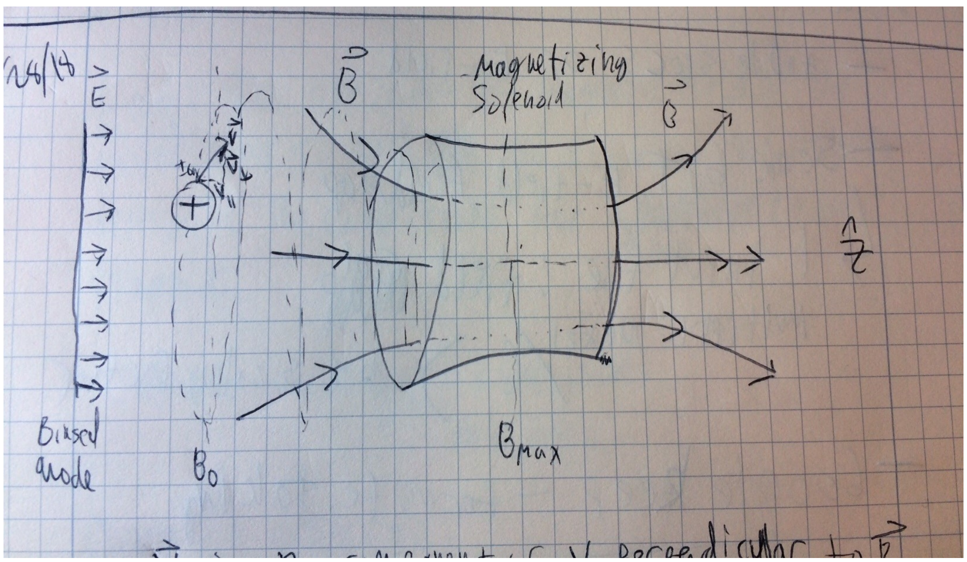

Figure 1: Diagram of an ion between the biased anode (left) and magnetizing solenoid (right).

We can setup a coordinate system with the z-axis going through the center of the anode and solenoid as shown in the figure. We'll start the derivation with two reasonable assumptions. First, we'll assume that anode is approximately a uniform charged disk of negligible thickness. Second, we'll assume that the ion is on the z-axis in order to simplify our derivations. In reality, the ion orbits the z-axis at some distance r. However, this distance is negligible compared to the radii of the anode and solenoid. Later, we'll consider how r grows with time and how valid this assumption becomes.

We'll start with the electric field from the anode. The electric field on the central axis (in this case, the z-axis) of a uniform disk of radius R_{anode} and surface charge density σ is given by:

$$\vec{E}_{anode} = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\left(r'\right)}{\left|\vec{r}-\vec{r}'\right|^3}\left(\vec{r}-\vec{r}'\right)d^2r' \tag{1}$$

where $\vec{r} = z\hat{z}$ denotes the location of the ion (in this case, along the z-axis) and $\vec{r}' = r'\hat{r} = r'\cos\theta\hat{x} + r'\sin\theta\hat{y}$ denotes the location of an element of surface charge on the anode. Note that by assumption, σ is constant and can be taken out of the integral. Thus,

$$\vec{r}-\vec{r}' = z\hat{z} - r'\cos\theta\hat{x} - r'\sin\theta\hat{y}$$
$$\left|\vec{r}-\vec{r}'\right| = \sqrt{z^2+r^2}$$

and so

$$\vec{E}_{anode} = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{R_{anode}} \frac{z\hat{z} - r'\cos\theta\hat{x} - r'\sin\theta\hat{y}}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} r'dr'd\theta \tag{2}$$

Note that $\int_0^{2\pi}\cos\theta d\theta = \int_0^{2\pi}\sin\theta d\theta = 0$, so only the term with $\hat{z}$ survives (as expected):

$$\begin{aligned}\vec{E}_{anode} &= \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{R_{anode}} \frac{z\hat{z}}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} r'dr'd\theta \\ &= \frac{\sigma}{4\pi\varepsilon_0}(2\pi)\, z\hat{z}\int_0^{R_{anode}} \frac{r'dr'}{\left(z^2 + r'^2\right)^{\frac{3}{2}}}\end{aligned}$$

Let $u = z^2 + r'^2$. Then $du = 2r'dr' \to \frac{1}{2}du = r'dr'$ and so

$$\begin{aligned}\vec{E}_{anode} &= \frac{\sigma z\hat{z}}{2\varepsilon_0}\int \frac{1}{2}u^{-\frac{3}{2}}du \\ &\to \frac{\sigma z\hat{z}}{2\varepsilon_0}\left(-u^{-\frac{1}{2}}\right) \\ &\to \frac{\sigma z\hat{z}}{2\varepsilon_0}\left(-\left(z^2 + r'^2\right)^{-\frac{1}{2}}\Big|_{r'=0}^{r'=R_{anode}}\right) \\ &= \frac{\sigma z\hat{z}}{2\varepsilon_0}\left(-\frac{1}{\sqrt{z^2 + R_{anode}^2}} + \frac{1}{z}\right) \\ &= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R_{anode}^2}}\right)\hat{z}\end{aligned} \tag{3}$$

The electric (Coulomb) force on an ion with charge q is given by

$$\vec{F}_{anode} = q\vec{E} = \frac{q\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R_{anode}^2}}\right)\hat{z} \tag{4}$$

We can now derive an equation for the magnetic field from the magnetizing solenoid. Since the solenoid is symmetric about the z-axis, the magnetic field will only depend on r and z:

$$\vec{B} = B_r\hat{r} + B_z\hat{z}$$

B_r can be derived from $\nabla \cdot \vec{B} = 0$ in cylindrical coordinates:

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} &= 0 \\ \frac{\partial}{\partial r}(rB_r) &= -r\frac{\partial B_z}{\partial z} \\ rB_r &= -\int_0^r r'\frac{\partial B_z}{\partial z}dr'\end{aligned}$$

If we assume that $\frac{\partial B_z}{\partial z}$ at $r = 0$ is known (defined) and does not change significantly in r, then we can assume to good approximation that it is constant and can be taken out of the integral:

$$\begin{aligned}rB_r &= -\frac{\partial B_z}{\partial z}\left(\frac{1}{2}r^2\right) \\ B_r &= -\frac{r}{2}\frac{\partial B_z}{\partial z}\end{aligned} \tag{5}$$

The magnetic force on a particle with charge q and moving with velocity $\vec{v}$ is

$$\begin{aligned}\vec{F} &= q\vec{v}\times\vec{B} \\ &= q\left[(v_\theta B_z - v_z B_\theta)\,\hat{r} - (v_r B_z - v_z B_r)\,\hat{\theta} + (v_r B_\theta - v_\theta B_r)\,\hat{z}\right]\end{aligned}$$

Since $B_\theta = 0$ by symmetry,

$$\vec{F} = q\left[v_\theta B_z \hat{r} + (v_z B_r - v_r B_z)\,\hat{\theta} - v_\theta B_r \hat{z}\right] \tag{6}$$

We are mainly concerned with F_z. Using (5), we have:

$$\begin{aligned}F_z &= -q v_\theta B_r \\ &= \frac{q r v_\theta}{2}\frac{\partial B_z}{\partial z}\end{aligned}$$

Averaging over one Larmor period, we have

$$F_{z,avg} = \mp \frac{q r_L v_\perp}{2}\frac{\partial B_z}{\partial z} \tag{7}$$

where we use the minus sign for positive ions. In order to determine $\frac{\partial B_z}{\partial z}$, we assume that the solenoid is thin compared to its length l (i.e. $R_{sol} \gg l$). In this case, the solenoid "looks like" a circular wire (or really N overlapping circular wires if the solenoid has N turns) of radius R_{sol} carrying current I. In this case, we can use the Biot-Savart law to determine the magnetic field:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l'} \times \hat{s}}{s^2} \tag{8}$$

where $\vec{s} = \vec{r} - \vec{r'}$. Since the ion is on the z-axis, we only need to consider the z component of $\vec{B}$. All other components cancel out. Thus,

$$B_z = \frac{\mu_0 I}{4\pi}\int \frac{dl' \cos\theta}{s^2} \tag{9}$$

where θ is the angle between $\vec{s}$ and $\vec{r'}$. Thus, $\cos\theta = \dfrac{R_{sol}}{\sqrt{R_{sol}^2 + z^2}}$, $s = \sqrt{R_{sol}^2 + z^2}$, and $\int dl'$ is the circumference of the wire, $2\pi R_{sol}$. Thus

$$\begin{aligned}B_z &= \frac{\mu_0 I}{4\pi}(2\pi R)\left(\frac{R_{sol}}{\sqrt{R_{sol}^2 + z^2}}\right)\left(\frac{1}{R_{sol}^2 + z^2}\right) \\ &= \frac{\mu_0 I}{2}\frac{R_{sol}^2}{(R_{sol}^2 + z^2)^{\frac{3}{2}}} \\ &= \frac{\mu_0}{4\pi}(2I)\frac{\pi R_{sol}^2}{(R_{sol}^2 + z^2)^{\frac{3}{2}}} \\ &= \frac{\mu_0}{4\pi}(2I)\frac{A_{sol}}{(R_{sol}^2 + z^2)^{\frac{3}{2}}}\end{aligned}$$

For a solenoid of N turns, we have

$$B_z = \frac{\mu_0}{4\pi}(2NI)\frac{A_{sol}}{(R_{sol}^2 + z^2)^{\frac{3}{2}}} = \tag{10}$$

Taking a partial derivative with respect to z:

$$\frac{\partial B_z}{\partial z} = \frac{\mu_0}{4\pi}(2NI)\frac{3A_{sol} z}{(R_{sol}^2 + z^2)^{\frac{5}{2}}}$$

Plugging this into (7) yields:

$$
\begin{aligned}
F_{z,avg} &= \mp \frac{q r_L v_\perp}{2} \frac{\partial B_z}{\partial z} \\
&= \mp \frac{q r_L v_\perp}{2} \left[\frac{\mu_0}{4\pi} (2NI) \frac{3 A_{sol} z}{(R_{sol}^2 + z^2)^{\frac{5}{2}}} \right] \\
\vec{F}_{sol} &= \mp (3 q r_L v_\perp) \left(\frac{\mu_0}{4\pi} \right) (A_{sol} N I) \frac{z}{(R_{sol}^2 + z^2)^{\frac{5}{2}}} \hat{z} \qquad (11)
\end{aligned}
$$

It is important to distinguish between the meaning of z in eqs. (4) and (11). In eq. (4), z is the distance along the z-axis from the center of the anode. In (11), it is the distance along the z-axis from the center of the solenoid. To be consistent with the experimental setup, let the origin ($z = 0$) be at the midpoint between the centers of the anode and solenoid such that the anode is positioned at $z = -l$ and the solenoid is at $z = +l$. With this setup, we can define the net force, $\vec{F}_{net} = \vec{F}_{anode} + \vec{F}_{solenoid}$, below

$$
\vec{F}_{net} = \frac{q\sigma}{2\varepsilon_0} \left(1 - \frac{(z+l)}{\sqrt{(z-l)^2 + R_{anode}^2}} \right) \hat{z} - (3 q r_L v_\perp) \left(\frac{\mu_0}{4\pi} \right) (A_{sol} N I) \frac{(z-l)}{\left(R_{sol}^2 + (z-l)^2 \right)^{\frac{5}{2}}} \hat{z}
$$

$$
\boxed{\vec{F}_{net} = \left[C_{anode} - C_{anode} \left(\frac{(z+l)}{\sqrt{R_{anode}^2 + (z-l)^2}} \right) - C_{sol} \left(\frac{(z-l)}{\left(R_{sol}^2 + (z-l)^2 \right)^{\frac{5}{2}}} \right) \right] \hat{z}}
$$

where we have condensed the constants into $C_{anode} = \dfrac{q\sigma}{2\varepsilon_0}$ and $C_{sol} = (3 q r_L v_\perp) \left(\dfrac{\mu_0}{4\pi} \right) (A_{sol} N I)$. To get a sense of what $\vec{F}_{net}$ looks like, we can sketch it using Mathematica with $C_{anode} = C_{sol} = l = 1$ and $R_{sol} = 2 R_{anode}$:

```
g[l_, b_, c_, z_] := 1 - (z + l) / Sqrt[b^2 + (z + l)^2] - (z - l) / ((c^2 + (z - l)^2)^(5 / 2))

Plot[g[1, 1, 2, z], {z, -5, 5}]
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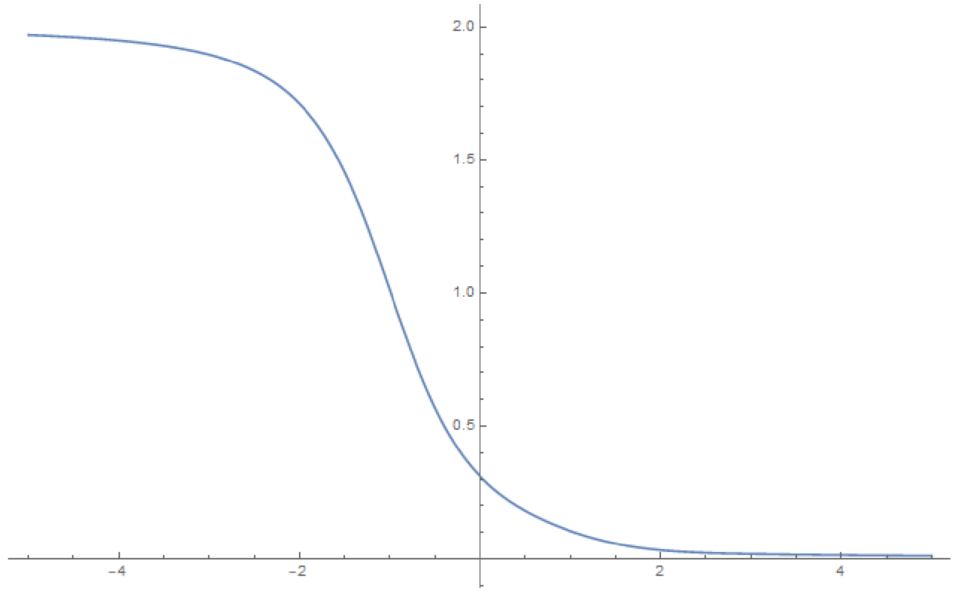

Figure 2: A sketch of the net force $\vec{F}_{net}$ as a function of z.

This plot should obviously be taken with a grain of salt: we would need to plug in realistic numbers for all of the constants to get a more accurate picture. That is, we need to know:

- Surface charge density of the anode σ
- Radii of the anode and solenoid R_{anode} and R_{sol}
- Distance between anode and solenoid $2l$ (known)
- Estimates for the larmor radius and angular (perpendicular) velocity r_L and $v_\perp$
- Cross-Sectional Area of the solenoid A_{sol} (derived from R_{sol})
- Number of turns of the solenoid N (known)
- Solenoidal current I (known)
- Constants q ($+e$ for ions), ε_0, and $\mu_0/4\pi$ (all are known)