Round-to-flat transformation of angular-momentum-dominated beams

Kwang-Je Kim
Advanced Photon Source, Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439, USA
(Received 13 June 2003; published 30 October 2003)

A study of round-to-flat configurations, and vice versa, of angular-momentum-dominated beams is presented. The beam propagation in an axial magnetic field is described in terms of the familiar Courant-Snyder formalism by using a rotating coordinate system. The discussion of the beam transformation is based on the general properties of a cylindrically symmetric beam matrix and the existence of two invariants for a symplectic transformation in 4D phase space.

DOI: 10.1103/PhysRevSTAB.6.104002 PACS numbers: 29.27.–a, 41.75.Lx, 41.85.–p

I. INTRODUCTION

Derbenev noted that the beam in transverse dimensions can be transformed from round to flat, or vice versa, by removing the correlation between the transverse degrees of freedom of an angular-momentum-dominated beam produced in an axial magnetic field [1]. Such a transformation may have applications for future linear colliders [2] or for the ultrafast x-ray generation [3]. The round-to-flat transformation has been experimentally demonstrated recently [4].

The fact that an initially round beam with a net angular momentum can be transformed to a beam asymmetric in transverse beam dimensions was first analyzed and experimentally demonstrated already in 1987 by a group working on ECR sources [5]. In this work, angular-momentum-dominated round beams were generated in an axial magnetic field and passed through a single quadrupole. The transverse dimensions of the beams then exhibit a large aspect ratio at a certain distance downstream. However, the correlation removal in this case is partial and transitory since only the correlation between the two transverse coordinates vanishes at a particular location. On the other hand, the Derbenev scheme, which requires at least three quadrupoles, achieves a complete removal of the correlation between the phase-space variables of two transverse dimensions. The resulting beams can then be characterized by two decoupled betatron motions with respective emittances.

The general theoretical framework for round-to-flat transformation was developed in Refs. [6,7]. The analyses in these papers are based on new sets of canonical variables—the guiding center variables in [6] and the coordinates for circular modes in [7]. The reason for introducing new variables appears to be the fact that the usual Cartesian canonical variables in the laboratory frame are not convenient to describe the helical motion occurring in angular-momentum-dominated beams. Although the formalisms involving the new variables in these papers are quite elegant, they are nevertheless not as straightforward as the familiar Courant-Snyder formalism based on the Cartesian coordinate system [8]. One of the two goals of this paper is to emphasize the well-known fact that the force due to an axial magnetic field in the laboratory frame appears as the force due to an axially symmetric, linearly focusing lens if viewed from a rotating coordinate frame [9,10]. The motion in the rotating frame, known as the Larmor frame, can be conveniently described in terms of the Cartesian canonical variables.

The second goal of this paper is to note that the discussion of the round-to-flat transformation can be greatly simplified by making use of two invariants of the beam matrix under symplectic transformation in 4D phase space. The first of these invariants, the volume of the 4D phase space, is well known as the 4D emittance. The second invariant is a trace of a combination of the beam matrix, first introduced in this form by Rangarajan *et al.* [11].

This paper is pedagogical in the sense that it is about the method of derivation rather than the results, which have been derived in previous papers, most extensively in Ref. [7]. Section II is a review of some general properties of a symplectic transformation in 4D phase space, in particular, the existence of two invariants. Section III presents a study of the general properties of a cylindrically symmetric beam matrix, in particular, the result that the general form of such a beam matrix takes a very simple structure in terms of the angular-momentum and 2D Courant-Snyder parameters. Section IV discusses the constraints on how a round beam may be transformed due to the existence of two invariants. We found that a round beam remains round if the beam angular momentum vanishes. The constraints also provide an expression for the ratio of emittances when a round beam with a nonvanishing angular momentum is transformed to an asymmetric beam. In Sec. V, an explicit model for round beam production and transport in an axial magnetic field is given. By using a rotating coordinate system, the motion can be described in terms of the well-known Courant-Snyder formalism. In Sec. VI, it is pointed out that an explicit realization of the round-to-flat transformation in the case of a nonvanishing initial emittance is found as a simple extension of the solution for the case of the

 1098-4402/03/6(10)/104002(6)$20.00

vanishing initial emittance. Finally, Sec. VII contains concluding remarks.

II. 4D SYMPLECTIC TRANSFORMATION AND TWO INVARIANTS

We consider particle distribution in 4D transverse phase space. The coordinates in this space are specified by a four-component vector. We find it convenient to represent the four-component vector in terms of two two-component vectors:

$$X = \begin{bmatrix} x \\ p_x \end{bmatrix}, \qquad Y = \begin{bmatrix} y \\ p_y \end{bmatrix}. \tag{1}$$

Here x and y are the positions in the x and y directions, respectively, and

$$(p_x, p_y) = \frac{p_s}{mc}(x', y') \equiv \frac{p_s}{mc}\left(\frac{d}{ds}x, \frac{d}{ds}y\right), \tag{2}$$

where p_s is the momentum in the axial direction, m is the particle mass, c is the speed of light, and s is the distance along the axial direction. As the beam is transported along an accelerator, the phase-space coordinates are transformed as

$$\begin{bmatrix} X \\ Y \end{bmatrix} \rightarrow M \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{3}$$

We will limit our discussion in this paper to the case where M is linear, and the motion is Hamiltonian. The M matrix is then symplectic:

$$\tilde{M} J_4 M = J_4. \tag{4}$$

Here $\sim$ denotes the transpose operation, and J_4 is the four-dimensional unit symplectic matrix

$$J_4 = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, \tag{5}$$

where we have introduced the 2×2 unit symplectic matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{6}$$

An extensive discussion of linear and nonlinear symplectic transformations can be found in [12], including the fact that if M is symplectic then so is $\tilde{M}$.

The global properties of a beam are described by beam moments. Assuming that the beam is centered properly, the first-order moments vanish:

$$\langle X \rangle = \langle Y \rangle = 0. \tag{7}$$

Here the angular brackets imply taking the average. The second-order beam moments can be organized into the 4×4 beam matrix:

$$\Sigma = \begin{bmatrix} \langle X\tilde{X} \rangle & \langle X\tilde{Y} \rangle \\ \langle Y\tilde{X} \rangle & \langle Y\tilde{Y} \rangle \end{bmatrix}. \tag{8}$$

Here, for example, $\langle X\tilde{Y} \rangle$ is the 2×2 matrix:

$$\langle X\tilde{Y} \rangle = \begin{bmatrix} \langle xy \rangle & \langle xp_y \rangle \\ \langle p_x y \rangle & \langle p_x p_y \rangle \end{bmatrix}. \tag{9}$$

The transformation, Eq. (3), induces the transformation of the beam matrix:

$$\Sigma \rightarrow M \Sigma \tilde{M}. \tag{10}$$

Because of the simplecticity of M, the transformation (10) leaves the following two quantities invariant:

$$\varepsilon_{4\mathrm{D}} = \det(\Sigma), \tag{11}$$

$$I_2(\Sigma) = -\frac{1}{2} T_R(J_4 \Sigma J_4 \Sigma). \tag{12}$$

In the above, det denotes the determinant and T_R the trace. The quantity $\varepsilon_{4\mathrm{D}}$ is well known and can be interpreted as the volume in 4D phase space. The trace invariant Eq. (12) was pointed out by Rangarajan *et al.* in the context of beam physics [11].

III. PROPERTIES OF ROUND BEAMS

The beam matrix Σ is symmetric and thus in general contains ten independent elements. In many cases of interest, however, beams are generated, accelerated, and transported in a cylindrically symmetric environment. The beam matrix must then be cylindrically symmetric:

$$\Sigma = M_R(\theta) \Sigma M_R^{-1}(\theta). \tag{13}$$

Here $M_R(\theta)$ is the matrix representing a rotation around the beam axis:

$$M_R(\theta) = \begin{bmatrix} I\cos\theta & I\sin\theta \\ -I\sin\theta & I\cos\theta \end{bmatrix}, \tag{14}$$

where I is the 2×2 unity matrix. By demanding that Eq. (13) be satisfied for an arbitrary θ, we obtain the following conditions:

$$\langle X\tilde{X} \rangle = \langle Y\tilde{Y} \rangle, \tag{15}$$

$$\langle X\tilde{Y} \rangle = -\langle Y\tilde{X} \rangle. \tag{16}$$

From Eq. (16), it follows:

$$\langle \widetilde{X\tilde{Y}} \rangle = -\langle \widetilde{Y\tilde{X}} \rangle = -\langle X\tilde{Y} \rangle. \tag{17}$$

Therefore the 2×2 matrix $\langle X\tilde{Y} \rangle$ is antisymmetric and can be written as

$$\langle X\tilde{Y} \rangle = \mathcal{L} J, \tag{18}$$

where J is the 2×2 unit symplectic matrix, Eq. (6).

The symmetric 2×2 matrix $\langle X\tilde{X} \rangle$ can be written as follows:

$$\langle X\tilde{X} \rangle = \varepsilon_{\mathrm{eff}} T_d, \tag{19}$$

where

$$T_d = D(d)T_0\tilde{D}(d), \tag{20}$$

$$D(d) = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}, \tag{21}$$

$$T_0 = \begin{bmatrix} \beta & 0 \\ 0 & 1/\beta \end{bmatrix}. \tag{22}$$

Here the quantity $\varepsilon_{\rm eff}$ and β can be interpreted as the emittance and the Courant-Snyder envelope function, respectively [8].

Collecting these results, we can write a cylindrically symmetric beam in the following form:

$$\Sigma = \begin{bmatrix} \varepsilon_{\rm eff}T_d & \mathcal{L}J \\ -\mathcal{L}J & \varepsilon_{\rm eff}T_d \end{bmatrix}. \tag{23}$$

With cylindrical symmetry, the beam matrix is thus greatly simplified, requiring only the parameters $\varepsilon_{\rm eff}$, β, $\mathcal{L}$, and d.

The quantity $\mathcal{L}$ is one-half of the angular momentum since

$$\langle xp_y - yp_x\rangle = 2\mathcal{L}. \tag{24}$$

However, it is not the kinetic angular momentum since it is measured in the rotating frame defined in Sec. V. It is the canonical angular momentum in the laboratory frame [10]. The canonical angular momentum is conserved in the presence of a cylindrically symmetric axial magnetic field. Experimental measurement of the beam angular momentum was discussed recently [13].

Equation (23) can also be written as

$$\Sigma = M_d\Sigma_0\tilde{M}_d, \tag{25}$$

where

$$M_d = \begin{bmatrix} D(d) & 0 \\ 0 & D(d) \end{bmatrix}, \tag{26}$$

$$\Sigma_0 = \begin{bmatrix} \varepsilon_{\rm eff}T_0 & \mathcal{L}J \\ -\mathcal{L}J & \varepsilon_{\rm eff}T_0 \end{bmatrix}. \tag{27}$$

In obtaining Eq. (25), we have used the relation

$$D_dJ\tilde{D}_d = J, \tag{28}$$

which follows from the symplecticity of D_d in x subspace. Equation (25) represents the translation to the location of the beam waist. In the following we will assume that the beam is at the waist since the translation to other locations is simple to perform.

A cylindrically symmetric beam is also said to be round in this paper.

IV. TRANSFORMATION OF ROUND BEAMS

To compute the two invariants, Eqs. (11) and (12), corresponding to the cylindrically symmetric Σ matrix, we start from the following identity:

$$JT_0J = -T_0^{-1}. \tag{29}$$

Using this identity it is straightforward to show

$$\Sigma_0\begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}\Sigma_0\begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} = -(\varepsilon_{\rm eff}^2 - \mathcal{L}^2)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{30}$$

By taking the determinant of both sides of this equation, it then follows:

$$\varepsilon_{\rm 4D} = \varepsilon_{\rm eff}^2 - \mathcal{L}^2. \tag{31}$$

Here we have introduced $\varepsilon_{\rm 4D} = \sqrt{\det(\Sigma_0)}$. Let us also introduce the thermal emittance $\varepsilon_{\rm th}$ by

$$\varepsilon_{\rm th} = (\varepsilon_{\rm 4D})^{1/2}. \tag{32}$$

Then

$$\varepsilon_{\rm eff} = \sqrt{\varepsilon_{\rm th}^2 + \mathcal{L}^2}. \tag{33}$$

Note that $\varepsilon_{\rm eff}$ is not a real emittance due to the correlation in the beam matrix.

The matrix product occurring in the left-hand side of Eq. (12) can be explicitly computed for Σ given by Eq. (25). From this, the trace invariant is found as

$$I_2(\Sigma) = 2(\varepsilon_{\rm eff}^2 + \mathcal{L}^2). \tag{34}$$

The existence of two invariants has important consequences. First, consider the case of the cylindrically symmetric channel. If a beam starts out to be cylindrically symmetric and is then transported through a cylindrically symmetric channel, the 4D emittance $\varepsilon_{\rm 4D}$ and the angular momentum must be separately conserved.

Next, let us now suppose that the cylindrically symmetric beam matrix Σ in Eq. (27) is diagonalized by a suitable simplectic transformation M:

$$M\Sigma\tilde{M} = \begin{bmatrix} \varepsilon_+T_+ & 0 \\ 0 & \varepsilon_-T_- \end{bmatrix}, \tag{35}$$

where $\varepsilon_\pm$ are constants and

$$T_\pm = \begin{bmatrix} \beta_\pm & 0 \\ 0 & 1/\beta_\pm \end{bmatrix}. \tag{36}$$

Note that the matrix M in Eq. (35) is in general not cylindrically symmetric. However, M is symplectic and thus the invariance of $\varepsilon_{\rm 4D}$ and I_2 is still valid. From the invariance of $\varepsilon_{\rm 4D}$,

$$\varepsilon_{\rm th}^2 = (\varepsilon_{\rm eff}^2 - \mathcal{L}^2) = \varepsilon_+\varepsilon_-. \tag{37}$$

From the invariance of I_2,

$$I_2 = 2(\varepsilon_{\rm eff}^2 + \mathcal{L}^2) = \varepsilon_+^2 + \varepsilon_-^2. \tag{38}$$

From these, it then follows:

$$\varepsilon_\pm = \varepsilon_{\rm eff} \pm \mathcal{L}. \tag{39}$$

Equation (39) is the main result of this section. It constrains the possibilities for manipulating a cylindrically symmetric beam via any symplectic transformation

M. If the angular momentum $\mathcal{L}$ vanishes, then

$$\varepsilon_{\rm th} = \varepsilon_+ = \varepsilon_-. \qquad (40)$$

That is, a round beam with vanishing angular momentum remains round under any symplectic transformation, even if the beam transport is highly nonsymmetric. This result was obtained earlier [14].

On the other hand, a round beam with nonvanishing angular momentum will become asymmetric if the X-Y correlation is removed by a suitable symplectic transformation, as was noted in the literature [1,5]. The emittances in the decoupled base are uniquely determined if the thermal emittance and the angular momentum are known, given by Eq. (39). When $\mathcal{L} \gg e_{\rm th}$, the beam is extremely asymmetric:

$$\frac{\varepsilon_+}{\varepsilon_-} \approx \left(\frac{2\mathcal{L}}{\varepsilon_{\rm th}}\right)^2, \qquad \mathcal{L} \gg \varepsilon_{\rm th}. \qquad (41)$$

Such a beam is said to be angular momentum dominated.

V. BEAMS IN AN AXIAL MAGNETIC FIELD

Beams with angular momentum are often produced and transported in an axial magnetic field. Particle motion in an axial magnetic field cannot be treated by the well-known Courant-Snyder formalism [8]. In Ref. [7], the usual Courant-Snyder formalism was generalized to describe the helical motion by introducing circular modes. In this paper we use a different approach based on the observation that the motion becomes uncoupled and simple in a rotating coordinate frame [9,10].

The rotation angle as a function of distance s along the axial direction is given by

$$\theta(s) = \int_0^s d\bar{s}\kappa(\bar{s}), \qquad (42)$$

where $\kappa(s)$ is the rate of rotation given by

$$\kappa(s) = \frac{B(s)}{2p_s}. \qquad (43)$$

Here $B(s)$ is the axial magnetic field, p_s is the particle momentum in the axial direction, and s is the distance in the axial direction. Let the coordinate vectors in the configuration space in the laboratory frame and in the rotating frame be $\boldsymbol{x}_L$ and $\boldsymbol{x}$, respectively. They are related by

$$\boldsymbol{x}_L \equiv \begin{bmatrix} x_L \\ y_L \end{bmatrix} = R(\theta)\begin{bmatrix} x \\ y \end{bmatrix} = R(\theta)\boldsymbol{x}. \qquad (44)$$

Here the 2×2 rotation matrix is

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \qquad (45)$$

Differentiating both sides of Eq. (44) with respect to s and then multiplying by p_s, we obtain

$$\boldsymbol{p}_L = R(\theta)(\boldsymbol{p} - \kappa_0 \boldsymbol{e}_s \boldsymbol{x}). \qquad (46)$$

In the above, $\boldsymbol{e}_s$ is the unit vector in the axial direction and

$$\kappa_0 = \frac{B(s)}{2mc}. \qquad (47)$$

The equations of motion in the rotating frame quantities $\boldsymbol{x}$ and $\boldsymbol{p}$ are obtained by inserting Eq. (46) into the usual laboratory frame equation of motion involving $\boldsymbol{x}_L$ and $\boldsymbol{p}_L$. The result is [10]

$$\left[\begin{array}{l} \boldsymbol{p} = \left[\begin{smallmatrix} p_x \\ p_y \end{smallmatrix}\right] = p_s \frac{d\boldsymbol{x}}{ds} = p_s \left[\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right], \\ \frac{d}{ds}\boldsymbol{p} = -p_s \kappa^2 \boldsymbol{x}. \end{array}\right. \qquad (48)$$

The motion represented by these equations is clearly Hamiltonian and uncoupled. Thus it can be described in terms of the usual Courant-Snyder formalism.

Consider the process of electron beam production from a cathode. An electron emitted from a cathode with phase-space coordinates $\boldsymbol{x}_L = (x_c, y_c)$, $\boldsymbol{p}_L = (p_{xc}, p_{yc})$ in the laboratory frame enters into a region with nonvanishing axial magnetic filed $B(s)$. The initial phase-space coordinates in the rotating frame are determined from Eqs. (44) and (46) by noting that the rotation matrix R is unity at the cathode surface $s = 0$:

$$\boldsymbol{x}|_{s=0} = \boldsymbol{x}_c, \qquad (49)$$

$$\boldsymbol{p}|_{s=0} = \boldsymbol{p}_c + \kappa_0 \boldsymbol{e}_s \boldsymbol{x}_c. \qquad (50)$$

In terms of the two-component phase-space vectors X and Y,

$$X|_{s=0} = \begin{bmatrix} x_c \\ p_{xc} - \kappa_0 y_c \end{bmatrix}, \qquad Y|_{s=0} = \begin{bmatrix} y_c \\ p_{yc} + \kappa_0 x_c \end{bmatrix}.$$

Thus, the phase-space coordinates at the cathode surface $s = 0$ in the rotating frame are related to those in the laboratory frame by the following linear transformation:

$$\begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix}_{s=0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\kappa_0 & 0 \\ 0 & 0 & 1 & 0 \\ \kappa_0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ p_{xc} \\ y_c \\ p_{yc} \end{bmatrix}. \qquad (52)$$

Note that this transformation is not symplectic. However, the transformation of the phase-space coordinates in the rotating frame from $s = 0$ to any point along a beam line is symplectic, since the motion in this region is governed by a Hamiltonian motion, Eq. (48).

For a cylindrically symmetric emission we can write

$$\langle x_c^2 \rangle = \langle y_c^2 \rangle = \sigma_c^2, \qquad (53)$$

$$\langle p_{xc}^2 \rangle = \langle p_{yc}^2 \rangle = \sigma_{pc}^2. \qquad (54)$$

It is also reasonable to assume that all correlation moments vanish:

$$\langle x_c y_c \rangle = \langle p_{cx} p_{cy} \rangle = \langle p_{cx} y_c \rangle = \langle p_{cy} x_c \rangle = \langle p_{cx} x_c \rangle = \langle p_{yc} y \rangle = 0. \tag{55}$$

The elements of the beam matrix at $s = 0$ are then easy to compute:

$$\Sigma|_{s=0} = \begin{bmatrix} \sigma_c^2 & 0 & 0 & \kappa_0 \sigma_c^2 \\ 0 & \sigma_{pc}^2 + \kappa_0^2 \sigma_c^2 & -\kappa_0 \sigma_c^2 & 0 \\ 0 & \kappa_0 \sigma_c^2 & \sigma_c^2 & 0 \\ -\kappa_0 \sigma_c^2 & 0 & 0 & \sigma_{pc}^2 + \kappa_0^2 \sigma_c^2 \end{bmatrix}. \tag{56}$$

This matrix is in the form of Eq. (27) with the identification

$$\varepsilon_{\text{eff}} = \sigma_c \sqrt{\sigma_{pc}^2 + \kappa_0^2 \sigma_c^2}, \tag{57}$$

$$\beta = \frac{\sigma_c}{\sqrt{\sigma_{pc}^2 + \kappa_0 \sigma_c^2}}. \tag{58}$$

$$\mathcal{L} = \kappa_0 \sigma_c^2. \tag{59}$$

From Eqs. (57) and (59), it follows that $\varepsilon_{\text{th}} = \sigma_c \sigma_{pc}$. The beam matrix after acceleration will in general have a larger thermal emittance $\sigma_c p_c$. However, the angular momentum will be strictly conserved if the channel is cylindrically symmetric.

VI. REMOVING *X-Y* CORRELATION: DIAGONALIZATION

The beam matrix in Eq. (23) contains correlation in the X-Y elements. A general procedure to remove the correlation by constructing the transformation matrix M was developed in Refs. [6,7] starting from the observation that the phase advances of the two transverse motions should differ from each other by 90°. An explicit procedure accomplishing this in terms of three quadrupoles was given by Edwards *et al.* in the case of vanishing thermal emittance [4]. Here we remark that it is straightforward to generalize the latter procedure to the case of nonvanishing thermal emittance.

Let A be the 2×2 matrix for (x, p_x) corresponding to a certain arrangement of quadrupoles and free space. The corresponding matrix B for (y, p_y) is obtained from A by replacing the quadrupole strength $q \to -q$. If the quadrupoles are rotated by 45°, then the 4×4 transformation matrix is

$$M = \frac{1}{2} \begin{bmatrix} A_+ & A_- \\ A_- & A_+ \end{bmatrix}, \tag{60}$$

where $A_\pm = A \pm B$. Assume that the matrix $M \Sigma_0 \tilde{M}$ is diagonal. The XY component of this matrix must vanish

$$0 = \varepsilon_{\text{eff}}(A_+ T_0 \tilde{A}_- + A_- T_0 \tilde{A}_+) + \mathcal{L}(A + J\tilde{A}_+ - A_- J\tilde{A}_-). \tag{61}$$

This equation is solved by

$$A_- = A_+ S, \tag{62}$$

where

$$S = \begin{pmatrix} 0 & -\beta \\ \frac{1}{\beta} & 0 \end{pmatrix}. \tag{63}$$

Here β is given by Eq. (59). Equation (62) can also be written as

$$A(1 - S) = B(1 + S). \tag{64}$$

This is a 2×2 matrix equation determining the matrices A and B. The determinants of both sides of Eq. (64) are identical from the properties of the matrices A, B, and S. Thus there are only three independent scalar equations. The solution can be constructed by three quadrupoles separated by free spaces, as was worked out for the case $\varepsilon_{\text{th}} = 0$ in Ref. [4].

Equation (62) was also derived in Ref. [7].

VII. CONCLUDING REMARKS

In this paper I presented a study of a round beam and its transformation to asymmetric beams by means of general properties of the symplectic transformation and the cylindrical symmetry. Although the formulas obtained here have all been derived previously, the method presented here may be useful in understanding round beam transformation.

ACKNOWLEDGMENTS

This paper was developed while discussing the flat beam experiment at the Fermilab/NICADD photoinjector laboratory [4]. The author thanks the participants of the experiment, in particular, Don Edwards, Helen Edwards, Steve Lidia, Philippe Piot, and Yin-e Sun for useful discussions. This work is supported by the U.S. Department of Energy, Office of Basic Energy Sciences, under Contract No. W-31-109-ENG-38.

[1] Ya. Derbenev, University of Michigan Report No. UM-HE-98-04, 1998.
[2] R. Brinkmann, Ya. Derbenev, and K. Floettmann, Phys. Rev. ST Accel. Beams **4**, 053501 (2001).
[3] LBNL Report No. LBNL-51766, 2002.
[4] D. Edwards *et al.*, in *Proceedings of 2001 Particle Accelerator Conference, Chicago, IL* (IEEE, Piscataway, NJ, 2001), pp. 73–75.
[5] H. L. Hagedoorn, H. Beuscher, R. K. Bhandar, J. I. M. Botman, W. Kraus-Vogt, and J. Reich, in *Proceedings of the International Conference on ECR Ion Sources and their Applications* (NSCL Report No. MSUCP-47, 1987); also J. Reich, H. Beuscher, R. K. Bhandari, J. I. M. Botman, and H. L. Hagedorn, in *Proceedings of the*

1988 European Particle Accelerator Conference, Stockholm, Sweden (IOP, Bristol, 1988), p. 833.
[6] A. Burov, S. Nagaitsev, A. Shemyakin, and Ya. Derbenev, Phys. Rev. ST Accel. Beams **3**, 094002 (2000).
[7] A. Burov, S. Nagaitsev, and Y. Derbenev, Phys. Rev. E **66**, 016503 (2002).
[8] E. D. Courant and H. S. Snyder, Ann. Phys. **3**, 1 (1958).
[9] A. J. Dragt, Nucl. Instrum. Methods Phys. Res., Sect. A **298**, 441 (1990).
[10] K.-J. Kim and C. X. Wang, Phys. Rev. Lett. **85**, 760 (2000).
[11] G. Rangarajan, F. Neri, and A. Dragt, in *Proceedings of the 1989 Particle Accelerator Conference, Chicago, IL* (IEEE, Piscataway, NJ, 1989), p. 1280.
[12] A. Dragt, in *Lectures on Orbit Dynamics*, edited by R. Carnigan *et al.*, AIP Conf. Proc. No. 87 (AIP, New York, 1982).
[13] Y. Sun *et al.*, in Proceedings of the PAC2003, Portland, OR (to be published).
[14] J. Rosenzweig and E. Colby, in *Proceedings of the 1993 Particle Accelerator Conference, Washington, D.C.* (IEEE, Piscataway, NJ, 1993), p. 3021.