Josh Yoskowitz
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Space Charge Benchmarking Derivation

# 1 Purpose

Determining distance along the $z$-axis that a $H_2^+$ ion crosses the $z$-axis. The ion starts 0.25m from the $z$-axis in the $y$-direction (its initial coordinates are $(0, 0.25, 0.5)$). The ion starts with some kinetic energy with its corresponding velocity in the $z$-direction. Along the $z$-axis, there is an electron current $\vec{I} = 1\text{A}\hat{z}$ through a distance of 1m along the $z$-axis. The ion is expected to be accelerated towards the $z$ axis due to space charge.

# 2 Derivation

Let $T_{ion}$ be the kinetic energy (to be calculated) of an ion at $(0, 0.25, 0.5)\,\text{m}$ with its corresponding initial velocity $\vec{v}_{ion} = v_{ion}\hat{z}$. The electric force $\vec{F}_e$ on the ion is perpendicular to its velocity, so the electric force (initially) acts as a centripital force $\vec{F}_c$. Solving for $v_{ion}$ yields:

$$\vec{F}_c = \vec{F}_e$$
$$\frac{m\left|\vec{v}_{ion}\right|^2}{R}(-\hat{z}) = qE(-\hat{z})$$
$$v_{ion} = \sqrt{\frac{qER}{m}}$$

where $R$ is the orbital radius. In this case, $R = 0.25$m initially. The magnitude of the electric field $E$ a distance $r$ away from a line of charge with charge density $\lambda$ is given by:

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}$$

In this case, $r = R$ initially. To calculate the charge density $\lambda$, we note that the electric current is made up of electrons uniformly distributed within a cylinder 1mm in radius with its central axis along the $z$-axis. The radius was selected to be small enough that we can approximate this as a line charge with uniform charge density $\lambda$. The electrons are moving with 130keV of kinetic energy in the $z$-direction. We can calculate the velocity of the electrons $v_e$:

$$\gamma = \frac{T_e}{m_e c^2} + 1$$
$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$
$$v_e = \beta c$$

Plugging in numbers, $m_e c^2 \approx 0.511\text{MeV}$, $\gamma = 1.254403$, $\beta = 0.603726$, $v_e = 1.8099 \times 10^8 \frac{\text{m}}{\text{s}}$. The electron current is on for 10ns and has 10nC of charge. In 10ns, the electrons travel $L = 1.809926358$m, so the charge density is

$$\lambda = \frac{Q}{L} = \frac{10\text{nC}}{1.809926358\text{m}} = 5.52508667\frac{\text{nC}}{\text{m}}$$

Thus, the electric field $E$ is

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} \approx 400\frac{\text{V}}{\text{m}}$$

And so the initial velocity of the ion is, with $q = e$, $R = 0.25$m, and $m_e$ being the mass of $H_2^+$,

$$v_{ion} = \sqrt{\frac{qER}{m}} \approx 68940\frac{\text{m}}{\text{s}}$$

The corresponding kinetic energy $T_{ion}$ is:

$$\beta = \frac{v_{ion}}{c} \approx 0.00230$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1.0000000264$$

$$T_{ion} = (\gamma - 1)\, m_{ion} c^2 \approx 50\text{eV}$$

We make the approximation that the electric force (and hence the acceleration in the $-\hat{y}$ direction) is roughly constant. In this case, we can calculate at which point the ion will cross the $z$-axis using kinetmatics. The time it takes the ion to reach the $z$-axis is given by:

$$t = \sqrt{\frac{2y}{a}} = \sqrt{\frac{2ym}{qE}} \approx 5.13\mu\text{s}$$

The velocity in the $z$-direction is constant, so the ion will travel $d = v_{ion} t \approx 0.354\text{m}$ from its initial starting $z$-position of 0.5m, so the ion will cross the $z$-axis at $\boxed{z \approx 0.854\text{m}}$. With the approximations, this is an over estimate, as the electric force increases as the ion approaches the $z$-axis. In the GPT simulation, the ion crosses at roughly $z = 0.812\text{m}$.