**6,7. Extrapolation of Asymmetry Data to Determine Ao**

*[including 10 graphs/figures and 5 data tables]*

**A. Extrapolation Functions**

The ultimate goal of a Mott asymmetry measurement is to provide an absolute value of the incident electron polarization, *Pe*. This is obtained by knowing the theoretical Sherman function *S*: *Pe* = *Ao/S*. Since *S* is calculated assuming elastic single-collision conditions, Ao corresponds to the Mott asymmetry taken when such conditions can be achieved experimentally. In principle, this requires that elastic scattering be guaranteed by energy filtering, and that a vanishingly thin target be used to eliminate the possibility of plural scattering. In practice one extrapolates measured asymmetries to zero target thickness, while providing the best possible energy discrimination against inelastically-scattered electrons [1]. At incident electron energies below ~200 keV, “retarding field” Mott polarimeters allow the precise extrapolation of asymmetries to zero energy loss in conjunction with target thickness extrapolations [2]. (It is important to note that energy extrapolation alone is not sufficient to guarantee single-scattering conditions! See reference [3], Figure 9.) At higher energies, where semiconductor or scintillator-based electron detection is used, energy discrimination becomes more difficult. In these cases, careful target thickness extrapolation procedures are mandatory.

In our experiment, we measured Mott asymmetries, *A(t)*, as a function of Au target foil thickness, *t*, ranging from 50 nm to 1000 nm. At 5 MeV in this foil thickness range, *A(t)* is a monotonically decreasing function of *t*, losing about 20% of its value as *t* increases from 0 nm (*Ao*) to 1000 nm. The function *A(t)* has a weak curvature with a positive second derivative. Historically, and because of the lack of any compelling theoretical guidance, a variety of functional forms have been used to fit *A(t)*, and thus determine *Ao* [3,4]. These have all been of the form

 $A^{q}\left(t\right)=A\_{o}(1-bt)$, (i)

 $A\left(t\right)=A\_{o}\frac{(1-bt)}{\left(1+at\right)} ,$ (ii)

or $A\left(t\right)=b+ae^{-ct},$ (iii)

where *q* = 1, -1, or -2, and *a*, *b*, *c*, and *Ao* are fitting parameters. In form (iii), *a*+*b* = *Ao* or, if *b* is set to zero, *Ao* = *a*. (The latter case has often given reasonable fits to asymmetry data for relatively thin foils, but implies the unphysical result that *A(∞)* = 0.) Form (iii) has been used only for incident energies below 200 keV, where, for the thickest targets, *A(t)* has a non-zero asymptotic value [3].

As we will see below, the precision with which *Ao* can be determined is limited in part by the statistical uncertainty in the *A(t)*, but mostly by the uncertainty in the target thicknesses. These are typically 5-10% of the *t* values themselves. An attractive alternative to thickness extrapolations is to consider *A* vs. the count rate summed from both detectors, *N(t)*. Since count rates in these experiments can be appreciable, the percentage uncertainty in *N*, 1/√*N*, can be much less than 10%. In this work, we will thus also consider *N*-dependent extrapolation functions.

The GEANT simulations discussed in Section X.X give us some confidence that a fitting form of type (i) with an additional quadratic term,

 $A\left(t\right)= A\_{o}(1-bt+at^{2})$, (iv)

is the most appropriate function with which to extrapolate our *A(t)* data to *Ao*. Having said this, we prefer a more conservative approach espoused in reference [4]. In their work, they fit their *A(t)* data to four first-order functions of types (i) and (ii), and found that the spread in their (correlated) fit values of Ao was somewhat larger than the statistical uncertainty in the Ao values given by a specific fitting form. As a result, they assigned an uncertainty to the weighted mean of the four intercepts (their quoted final value of *Ao*) to be such that ±2σ error bars encompassed all four intercepts.

To this end, we have applied a more general procedure to assess the precision of our final *Ao* values. Our *A(t)* data were fit using the method of Padé approximates [5]. Padé approximates (PAs) are a class of rational fractions which are typically well-behaved and converge more rapidly than Taylor series approximations to a set of data for extrapolation. The PAs (*n,m*) take the form

 $A(t)=\frac{A\_{o}(a\_{n}t^{n}+a\_{n-1}t^{n-1}+…+a\_{2}t^{2}+a\_{1}t+a\_{0})}{(b\_{m}t^{m}+b\_{m-1}t^{m-1}+…+ b\_{2}t^{2}+b\_{1}t+b\_{0})}$ (v)

for *m* ≥ 0 and *n* ≥ 1. The form of eq.(i) thus corresponds to a (1,0) PA for *q* = 1, (0,1) for *q* = -1, and a (0,2) PA for *q* = -2; equations (ii) and (iv) correspond to a PA of (1,1) and (2,0), respectively. Finally, equation (iii) is essentially a PA of arbitrarily high order *s* of the form (*s*,0).

We began our analysis by using the (1,0) form to fit a given *A(t)* data set, and then increasing both *n* and *m* until application of an F test indicated that higher orders of n and/or m were not justified [6]. All fits that passed the F-test were also subjected to a reduced chi-squared analysis as well [6]. As we will show below, the only PA forms that provided valid fits to the *A(t)* data were the (1,0), (2,0), (0,1), and (1,1) forms. This procedure was repeated for fits to the *A(N)* data sets. In this case, only the (1,1), (0,2), and (2,0) forms proved valid.

**B. Cuts to the Data Prior to Fitting**

Several cuts were made to the raw scintillator count data to yield the *A(t)* and *N(t)* values that were fit as discussed above. First, as discussed in more detail previously in Section X.X, a YY ns timing window, triggered on the beam pulse architecture, was set up corresponding to simple elastic scattering from the target directly into the main scintillators. Scintillator counts that occurred before or after this window were discarded. The subsequent scintillator pulse-height spectra are shown in Figure A.

 

**FIGURE A**

Dominating this spectrum is the quasi-elastic peak. This peak comprises the events from which we wish to construct our asymmetries, and is fit well (?) with a Gaussian function, indicated by the green curve. From these events, we next subtract the “dilution” background, indicated by the magenta fit line. These counts are due largely to inelastic scattering events in the target, scintillators, and the various metallic mechanical apertures and walls in the apparatus. They are fit nicely by a decreasing exponential.

The cuts just described represent a crude energy filtering of the scintillator events; we now wish to consider more carefully any residual depolarizing inelasticity that may exist across the remaining Gaussian pulse-height spectrum. This can be accomplished by considering sequential slices in pulse height across the quasi elastic Gaussian profile shown in Figure A [7], and calculating the Mott asymmetry of a given slice as a function of the number of standard deviations of the Gaussian the slice is from the distribution’s mean. We show such a result in Figure B for one of the 50 nm foils.

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**FIGURE B**

Such plots were made for all foil thicknesses, and exhibited similar asymmetry-pulse-height dependence. One would expect that purely elastic events would yield the highest asymmetry, with *A* values falling off both to the right and left of the Gaussian, corresponding to the unsubtracted high-pulse-height tail of the inelastic background and small-energy-loss target scattering events, respectively. Based on this analysis, we decided to restrict the counts used to construct *A(t)* and *N(t)* to those within -1/2*σ* and +2*σ* of the Gaussian mean. This procedure did not significantly reduce the statistical precision of our results. Another analysis we performed to check these cuts was to plot *Ao* values for various fitting functions when individual *A(t)* values were calculated from various pulse-height subsections of the Gaussian. This yielded a plot essentially equivalent to that shown in Figure B.

C. **Uncertainty of the Data**

The uncertainty of *Ao* for an extrapolated fit to *A(t)* or *A(N)* is determined solely by uncertainty in the *t* or *N* values. Since one of the goals of this work was to try to measure *Ao* values with a precision of better than 0.5%, we generally counted long enough to achieve statistical precisions for *A(t)* < 0.2% of the *A* value itself. This typically yielded better than 0.2% percentage precision on values of *N*.

Gold target thicknesses were measured using a field emission scanning electron microscope (FESEM) technique [8]. A gold foil that was manufactured in the same batch as the target foils was mounted to a silicon substrate that was then cleaved to expose a thickness cross section for SEM imaging. The images were typically made at a single location for each sample prepared. For two of the samples, to test uniformity across a small area of the foil, the sample was translated and 3 or 4 spots were measured along the edge of a sample. Additionally, for two foils, two FESEM samples were prepared from a single target foil, one near the center and one near the edge, and both were analyzed. Finally, the tilt (pitch) dependence of the mounting in the FESEM was studied for one foil.

Statistical and systematic uncertainties in the foil thickness measurements can come from a number of sources. These are summarized in Table A. Statistical uncertainties include the variation in the thickness measurement obtained from images of the same spot on a film, which may vary in magnification or working distance. Additionally, repeated analysis of the same FESEM image using different software algorithms yielded a distribution of thickness values. These are quantified in Table A, row 3. [Merge rows 3,4]

Systematic uncertainties in the foil measurements result from inherent limitations of the FESEM machine, uncertainties introduced in the image analysis for foil thickness calculation, and uncertainties introduced through the foil mounting technique. The inherent resolution of the FESEM is 1.2 nm; uncertainty in foil tilt contributes a percentage uncertainty of 0.4%. Instrument focus resolution added another 1% uncertainty. These combined uncertainties are listed in row 4 of Table A. [Combine first 4 “syst.” Rows of Table A.]

Row 5 of Table A gives the manufacturer’s specifications for the thickness variation across a single foil. Translation studies, where a sample was measured at different positions along its surface, and larger studies where different FESEM samples were prepared from near the edge and the center of a gold sibling foil, were consistent within the resolution of the measurements with the manufacturer’s assertion that the FESEM thickness varies by no more than 2% across the sample [9]. The target foils used in this study were manufactured in the same batch as the samples mounted for FESEM measurements, but the manufacturer guarantees only that these siblings are consistent within 5%. This comprises the largest systematic uncertainty in the target foil measurement (Row 6 of Table A).

Thickness uncertainties are associated with the abscissa in a plot of *A* vs. *t*. These uncertainties, given in the last row of Table A, were converted to ordinate errors by multiplying them by the local slope of any given fit at the corresponding thickness values. Thus our reported final fit values represent an iterative procedure in which the data are first fit by ignoring the abscissa errors. Then the ordinate uncertainties due to thickness uncertainty are determined using the preliminary fit, and added in quadrature with the ordinate statistical counting uncertainties. A final fit is then generated. We note that the ultimate uncertainty thus obtained for *Ao* is completely dominated by the foil thickness uncertainty. An equivalent procedure was used to fit the *A* vs. *N* data, but in this case, the (now correlated) uncertainties in *A* and *N* had comparable influence on the final uncertainty in *Ao*.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | *nominal thickness (nm)* | *1000* | *870* | *750* | *625* | *500* | *355* | *225* | *50* |
|  | **mean thickness (all data, nm)** | **943.7** | **836.8** | **774.6** | **561.2** | **482.0** | **389.4** | **215.2** | **52.0** |
| Stat. | Stdev, nom. identical data (nm) | 29.0 | 7.1 | 9.1 | 8.0 | 9.7 | 4.5 | 1.9 | 2.3 |
|  | stdev image reanalysis (nm) | 22.5 | 7.7 | 9.4 | 7.5 | 4.0 | 2.7 | 1.8 | 2.1 |
| Syst. | Image analysis: ± 4 Pixel  | 20.0 | 8.0 | 10.0 | 8.0 | 8.0 | 8.0 | 2.6 | 2.6 |
|  | Resolution (1.2 nm inherent) | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 |
|  | Tilt (0.4%) | 4.6 | 4.2 | 3.9 | 2.8 | 2.5 | 1.9 | 1.1 | 0.3 |
|  | Focus (1%) | 9.4 | 8.4 | 7.7 | 5.6 | 4.8 | 3.9 | 2.2 | 0.5 |
|  | Different spots (Lebow: 2%) | 18.9 | 16.7 | 15.5 | 11.2 | 9.6 | 7.8 | 4.3 | 1.0 |
|  | Sibling difference (Lebow:5%) | 47.2 | 41.8 | 38.7 | 28.1 | 24.1 | 19.5 | 10.8 | 2.6 |
| Totals |  |  |  |  |  |  |  |  |  |
|  | stat uncertainty (nm) | 36.7 | 10.5 | 13.1 | 11.0 | 10.5 | 5.2 | 2.6 | 3.1 |
|  | syst uncertainty (nm) | 55.6 | 46.7 | 43.8 | 31.9 | 27.7 | 22.9 | 12.2 | 4.1 |
|  | **total uncertainty (nm)** | **66.6** | **47.9** | **45.7** | **33.7** | **29.6** | **23.5** | **12.5** | **5.1** |

**TABLE A**

**D. Results**

Table B lists the data taken for Run 1 (January 2015) and Run 2 (October 2015; Section FFF) for *A(t)* and *N(t)* and their respective uncertainties. In order to check the reproducibility of the data, twin 50 nm and 1000 nm foils were used. Moreover, a specific 1000 nm foil target was used both at the beginning and at the end of each Run to check for overall drifting in time. Figures C – H show these data fit to the PA forms suggested by our GEANT simulations (Section GGG): (2,0) for *A(t)*, (C,D) for *N(t)*, and (T,H) for *A(N)*.

Tables C,D, and E contain the fitting parameters for a variety of PA forms to the *A(t)*, *N(t)*, and *A(N)* data showing also the F-test and Χ2ν results. These PA functions begin with the simplest forms, (0,1) and (1,0), and increase in complexity through the first ones to be rejected due to the F-test or unlikely values of Χ2ν.  We note that the GEANT-suggested forms for *A(t),* *N(t)*, and *A(N),* (2,0), (C,D), and (T,H), respectively, meet the F-test and Χ2ν criteria.

Figure I. These need to be redone with corrected values.

The (1,0) function appears to be a poor fit to *A(t)* by eye. However, it cannot be excluded based on an F test since it has no lower-order function with which it can be compared, nor can it be rejected outright based on the fit’s Χ2ν value (?? though 2.38 and 1.85 for a 9 degree of freedom fit have 95 and 99% associated in the chart – any way to exclude based on that?). To investigate the dependence of this fit on the number of target foils used in the data set, successive foils were removed from the fit starting with those that were 1000 nm thick. Indeed, *Ao* for the (1,0) fit increases with each foil removed toward a value more consistent with the other values, but when removing successive target foils from the data set, the other functions approximating the data also have deviations upward when, in particular, the foil set is reduced to the thinnest five and six data points. These results are shown in Figure I. The tightest bunching of the *Ao* results for all of these fits come when the 870 and 1000 nm foil data are discarded. However, there are no significant differences in the statistical differences of these (correlated) *Ao* intercepts for a given PA form as we successively remove all targets thicker than 500 nm. Based on these considerations, we have elected to use only foils with thicknesses less than 600 nm for our final results.

Figures J – M show the results of the accepted fits for the reduced foil sets in histogram form for both *A(t)* and *A(N)* in Runs 1 and 2. There is no evidence between the Runs for a systematic variation of the polarization over the 9-month hiatus between the runs. This is somewhat surprising, given that the photocathode was reactivated several times during this period. There is good statistical agreement between the *Ao* for extrapolations based on rate and thickness. We conclude that the ultimate precision of this device is 0.3% of the *Ao* value itself.

**Figure J.** Distribution of the values for *Ao* extrapolated using the non-excluded Pade(n,m) terms for Run 1 for *A(t)*.

**Figure K.** Distribution of the values for *Ao* extrapolated using the non-excluded Pade(n,m) terms for Run 2 for *A(t)*.

**Figure L.** Distribution of the values for *Ao* extrapolated using the non-excluded Pade(n,m) terms for Run 1 for *A(N)*.

**Figure M.** Distribution of the values for *Ao* extrapolated using the non-excluded Pade(n,m) terms for Run 2 for *A(N)*.

 **References**

1. T.J. Gay and F.B. Dunning, “Mott Electron Polarimetry,” Rev. Sci. Instrum. **63**, 1635 (1992).

2. Hodge et al. RSI 1979.

3. T.J. Gay, J.A. Brand, J.E. Furst, M.A. Khakoo, W.V. Meyer, W.M.K.P. Wijayaratna, and F.B. Dunning, “Extrapolation Procedures in Mott Electron Polarimetry,” Rev. Sci. Instrum. **63**, 114 (1992).

4. G.D. Fletcher, T.J. Gay, and M.S. Lubell, “New Insights Into Mott–Scattering Electron Polarimetry,” Phys. Rev. A **34**, 911 (1986).

5. Reference for Padé approximant method.

6. Bevington, 2nd ed.

7. Ladish?

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9. Lebow Company, XXXXXXXXX

**Need:**

N vs. t plots

Remind Joe to discuss the overview of Runs 1 and 2

What does GEANT recommend for A vs. N and N vs. t?

In Table A merge rows 3 and 4, and 5 - 8