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Analyzing electron beam using BPM from "Lecture Notes on Topics in Accelerator Physics, Section 4: Fast Ion Instability" by Alex Chao, November 2002 SLAC-PUB-9574

Assume we have a beam position monitor (BPM) located at $s = 0$ along the accelerator. Let an electron bunch of length l circulate in a storage ring of circumference $C = cT_0$. Since the beam is bunched, the BPM will measure an alternating current signal via pick-up electrodes. Based on the amplitudes of the signals on the electrodes, the position of the beam can be determined. The time-dependent signal seen by the BPM is

$$\text{signal}(t) = \sum_{k=0}^{\infty} y_e(kC|ct - kC)|_{0<ct-kC<l}$$

where y_e is the transverse distance of an electron from the beam centroid and k sums over multiple turns. We can take a Fourier transform of the BPM signal into frequency space:

$$\begin{aligned}
\text{spectrum}(\Omega) &\propto \int_0^{\infty} dt e^{-i\Omega t}\text{signal}(t) \\
&= \sum_{k=0}^{\infty} \int_0^{l/c} dt' e^{-i\Omega\left(t'+kT_0\right)} y_e(kC|ct') \\
&= \sum_{k=0}^{\infty} e^{-i(\Omega+\omega_\beta)kT_0} \int_0^{l/c} dt' e^{-i(\Omega-\omega_I)t'} \tilde{y}_e(kC|ct') \\
&= y_0 \sum_{k=0}^{\infty} e^{-i(\Omega+\omega_\beta)kT_0} \int_0^{l/c} dt' e^{-i(\Omega-\omega_I)t'} \frac{e^{\eta'}}{\sqrt{2\pi\eta'}}
\end{aligned}$$

where $\eta' = t'\sqrt{K\omega_I kC/2\omega_\beta l}$. From the second to third step, the form of y_e was used:

$$y_e(s|z) = \tilde{y}_e(s|z)\, e^{-i\omega_\beta s/c + i\omega_I z/c}$$

From the third to the fourth step, the form of $\tilde{y}_e$ in the asymptotic regime ($\eta \gg 1$) was used:

$$\tilde{y}_e(s|z) = y_0 I_0(\eta) \approx y_0 \frac{e^{\eta}}{\sqrt{2\pi\eta}}$$

The integral in the last step is of the form:

$$I = \int_0^{l/c} dt' \frac{e^{(B-iA)t'}}{\sqrt{2\pi B t'}} = \frac{1}{\sqrt{2\pi B}} (-B+iA)^{-\frac{1}{2}} \gamma\left(\frac{1}{2}, (-B+iA)\frac{l}{c}\right)$$

where $A = \Omega - \omega_I$ and $B = \sqrt{K\omega_I kC/2\omega_\beta l}$ and $\gamma(\alpha, x)$ is the lower incomplete Gamma function:

$$\gamma(\alpha, x) = \int_0^{x} t^{\alpha-1} e^{-t} dt$$

For $|x| \gg 1$, $\gamma(\alpha, x) \approx -x^{\alpha-1}e^{-x}$. Thus, with $|A|\, l/c \gg Bl/c \gg 1$ (from the validity criterion $\frac{\omega_I l}{c} \gg \eta \gg 1$ with $\eta = \frac{z}{c}\sqrt{\frac{K\omega_I s}{2\omega_\beta l}}$),

$$I \approx \sqrt{\frac{l/c}{2\pi B}} e^{Bl/c} \left(\frac{e^{-iAl/2c}\sin\frac{Al}{2c}}{Al/2c}\right)$$

We can then plug this into the equation for spectrum (Ω). If we measure the signal in a small window around a large $k = \overline{k}$ (note that the signal obviously diverges for $k \to \infty$), we have

$$|\text{spectrum}(\Omega)| \propto y_0 \sqrt{\frac{l/c}{2\pi \overline{B}}} e^{\overline{B}l/c} \left| \frac{\sin \frac{(\Omega-\omega_I)l}{2c}}{(\Omega - \omega_I)\, l/2c} \right| \sum_{p=-\infty}^{\infty} \delta\left(\Omega + \omega_\beta - p\omega_0\right)$$

where $\omega_0 = 2\pi/T_0$ is the revolution angular frequency, $\overline{B} = \sqrt{K\omega_I \overline{k} C/2\omega_\beta l}$ (similar to η above), and we have plugged in $A = \Omega - \omega_I$. We see that the electron beam spectrum contains δ-function peaks at $\Omega = p\omega_0 - \omega_\beta$ corresponding to the lower betatron sidebands of all revolution harmonics. It also contains a broad envelope $\frac{\sin\left[(\Omega - \omega_I)\, l/2c\right]}{(\Omega - \omega_I)\, l/2c}$ around $\Omega = \omega_I$, the characteristic ion frequency, with width $\Delta\Omega \pm \pi c/l$. Thus, for longer bunches, the width of the envelope decreases and becomes more defined. The entire spectrum also grows with time according to the factor $e^{\overline{B}l/c}/\sqrt{2\pi \overline{B}l/c}$, as we would expect.

To see what this spectrum looks like, we can plot $|\text{spectrum}(\Omega)|$ as a function of Ω for several cases. Let's plug in some reasonable numbers: $\omega_\beta = 5\text{MHz}$, $K = \frac{4\Sigma n N c^2 r_e}{\gamma a^2}, 6.5 \times 10^7 \text{s}^{-2}$, $C = cT_0 = \frac{2\pi c}{\omega_0} = 2000\text{m}$, $T_0 = C/c = 6.7\mu\text{s}$, and $\omega_0 \approx 0.942\text{MHz}$ (the revolution angular frequency). We'll make plots of $\frac{|\text{spectrum}(\Omega)|}{y_0}$ vs. Ω for various values for $\bar{k}$, l, and ω_I. We'll consider the characteristic frequencies $\omega_I = \sqrt{\frac{2Nr_p c^2}{la^2 A}}$ for H_2^+, CH_4^+, N_2^+,and CO_2^+ with $N = 10^{11}$ electrons, $a = 1\text{mm}$, $r_p = 1.54 \times 10^{-16}\text{cm}$, and $A = \frac{M}{m_p}$ ($A_{H_2^+} = 2$, $A_{CH_4^+} = 16$, $A_{N_2^+} = 28$, $A_{CO_2^+} = 44$). Below is a table of parameters/calculated values for each plot:

Plot	$\bar{k}$	ω_I(Hz)	l(m)	$\bar{B}$(Hz)
1	10^1	1.18×10^8 (H_2^+)	1	3.91×10^6
2	10^1	4.16×10^7 (CH_4^+)	1	2.32×10^6
3	10^1	3.15×10^7 (N_2^+)	1	2.02×10^6
4	10^1	2.51×10^7 (CO_2^+)	1	1.81×10^6
5	10^2	1.18×10^8 (H_2^+)	1	1.24×10^7
6	10^4	1.18×10^8 (H_2^+)	1	1.24×10^8
7	10^1	3.72×10^7 (H_2^+)	10	6.96×10^5
8	10^1	1.18×10^7 (H_2^+)	100	1.24×10^5
9	10^1	4.16×10^6 (CH_4^+)	100	7.36×10^4
10	10^1	3.15×10^6 (N_2^+)	100	6.40×10^4
11	10^1	2.51×10^6 (CO_2^+)	100	5.71×10^4
12	10^1	All Four ω_I	100	N/A

Table 1: Calculated values for $\overline{B}$ for various $\overline{k}$

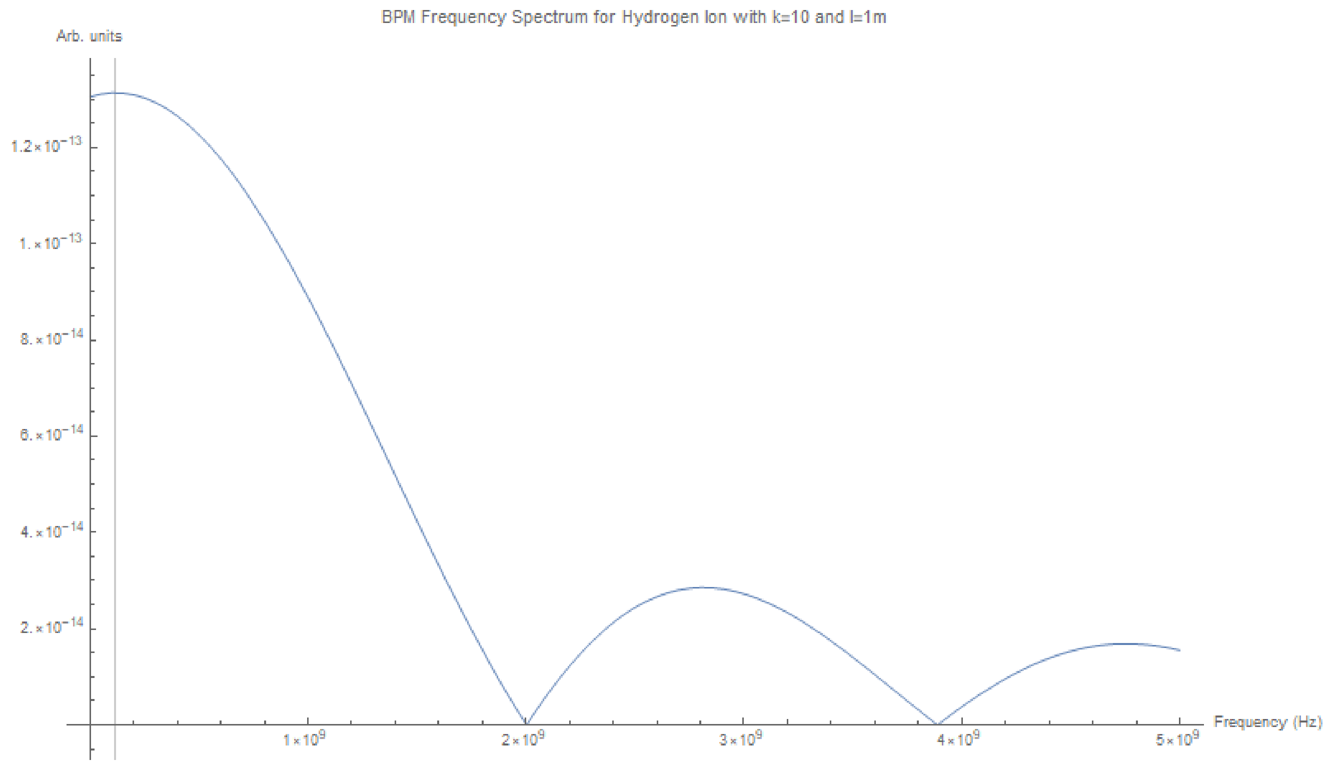

Figure 1: BPM Frequency Spectrum 1: H_2^+ with $k = 10$, and $l = 1$m

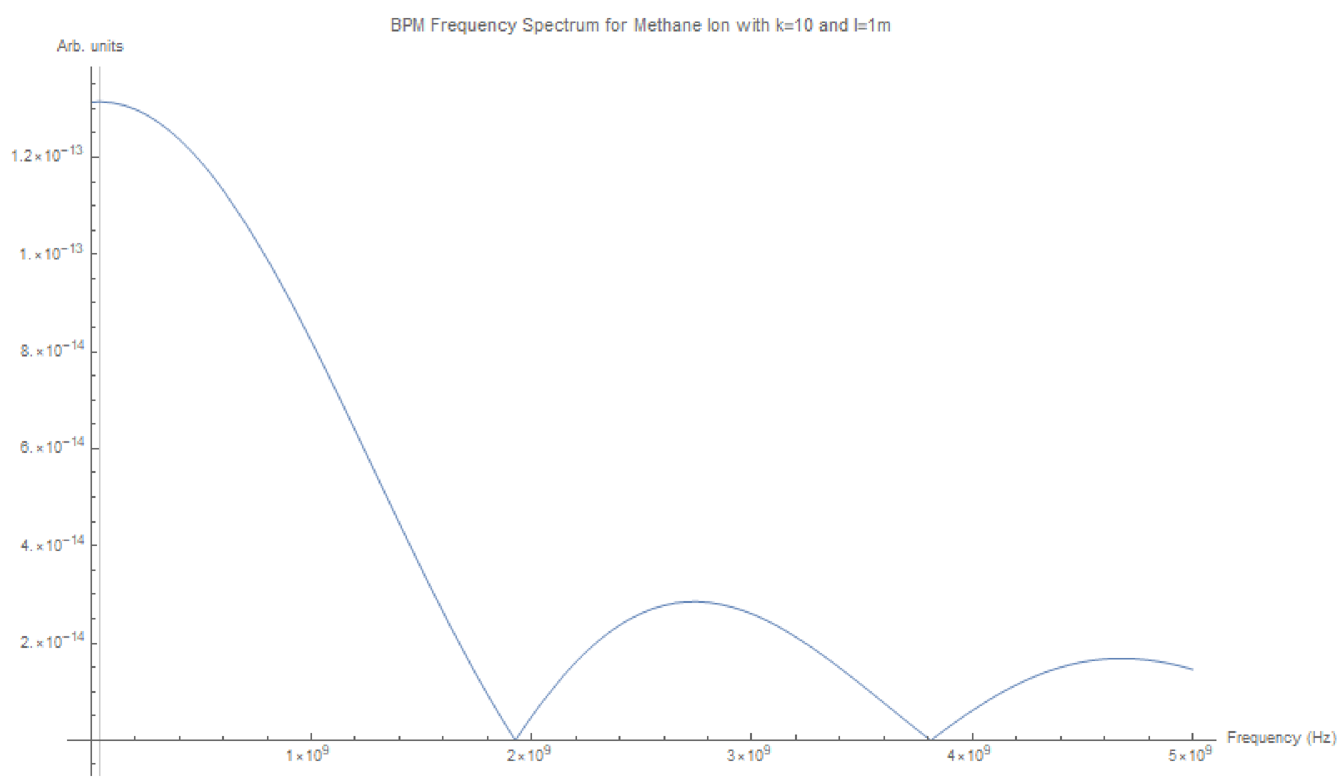

Figure 2: BPM Frequency Spectrum 2: CH_4^+ with $k = 10$ and $l = 1$m

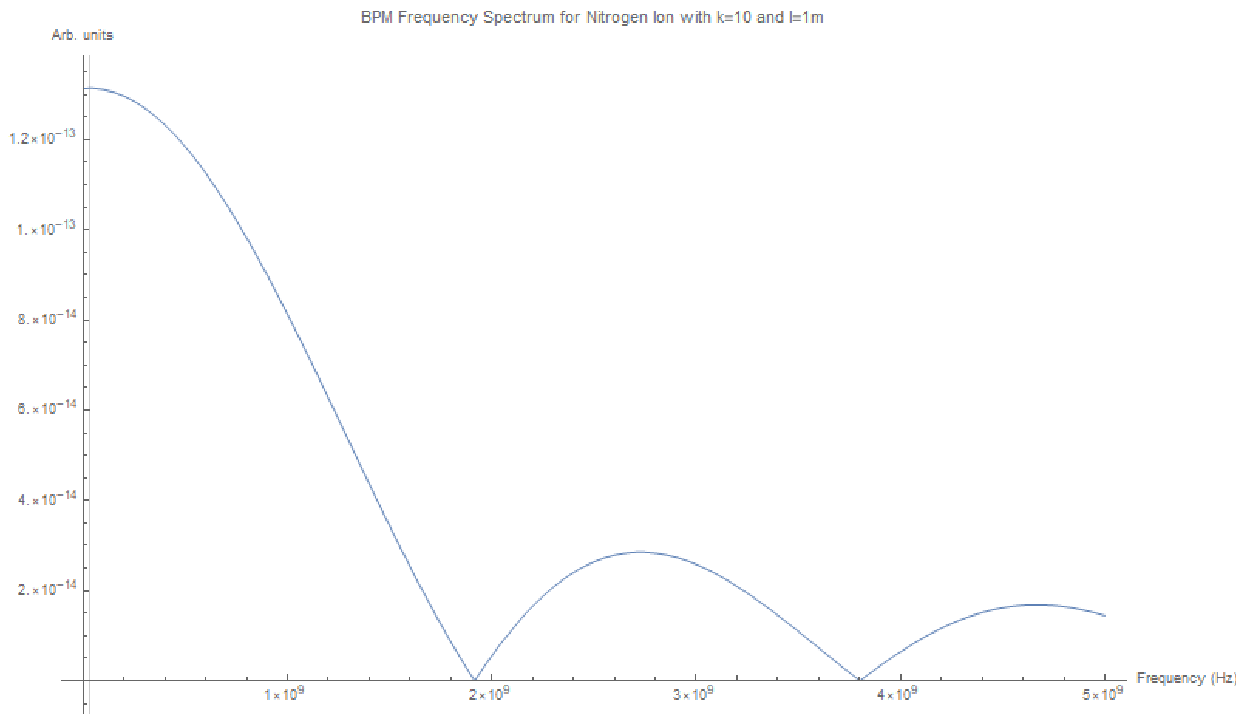

Figure 3: BPM Frequency Spectrum 3: N_2^+ with $k = 10$ and $l = 1$m

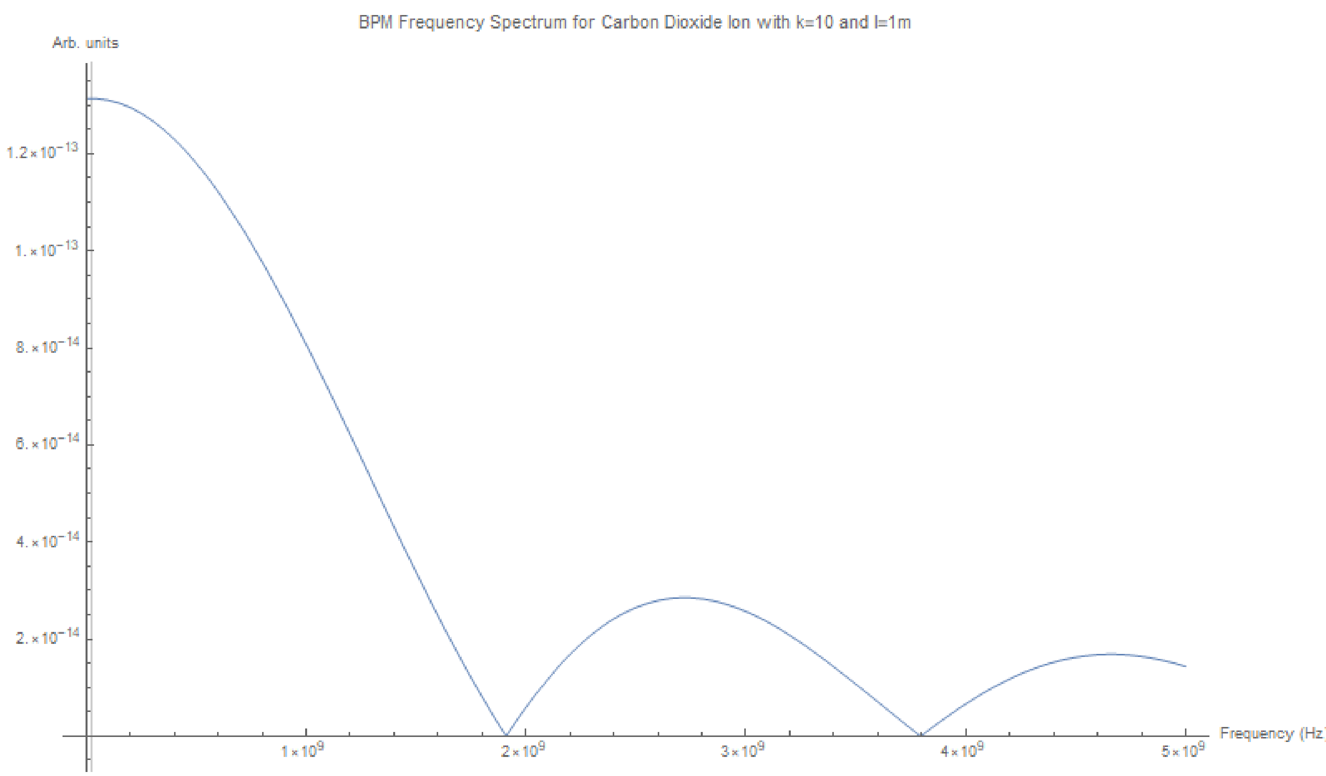

Figure 4: BPM Frequency Spectrum 4: CO_2^+ with $k = 10$ and $l = 1$m

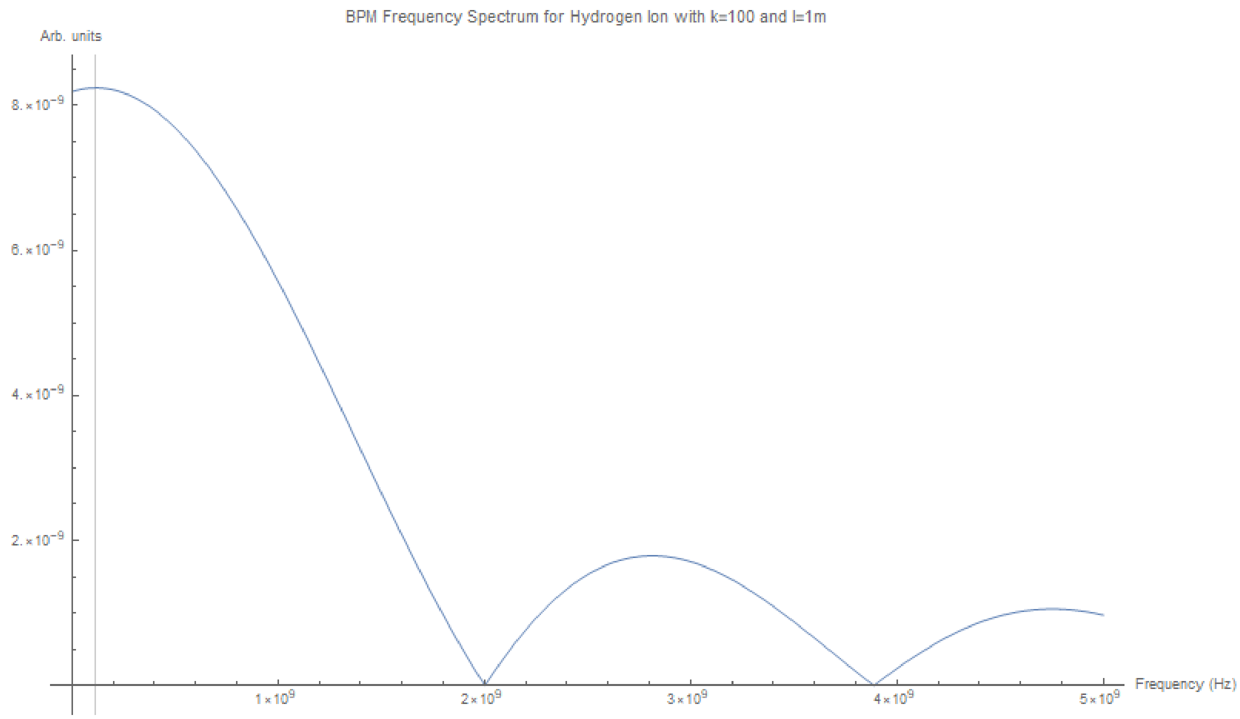

Figure 5: BPM Frequency Spectrum 5: H_2^+ with $k = 100$ and $l = 1$m

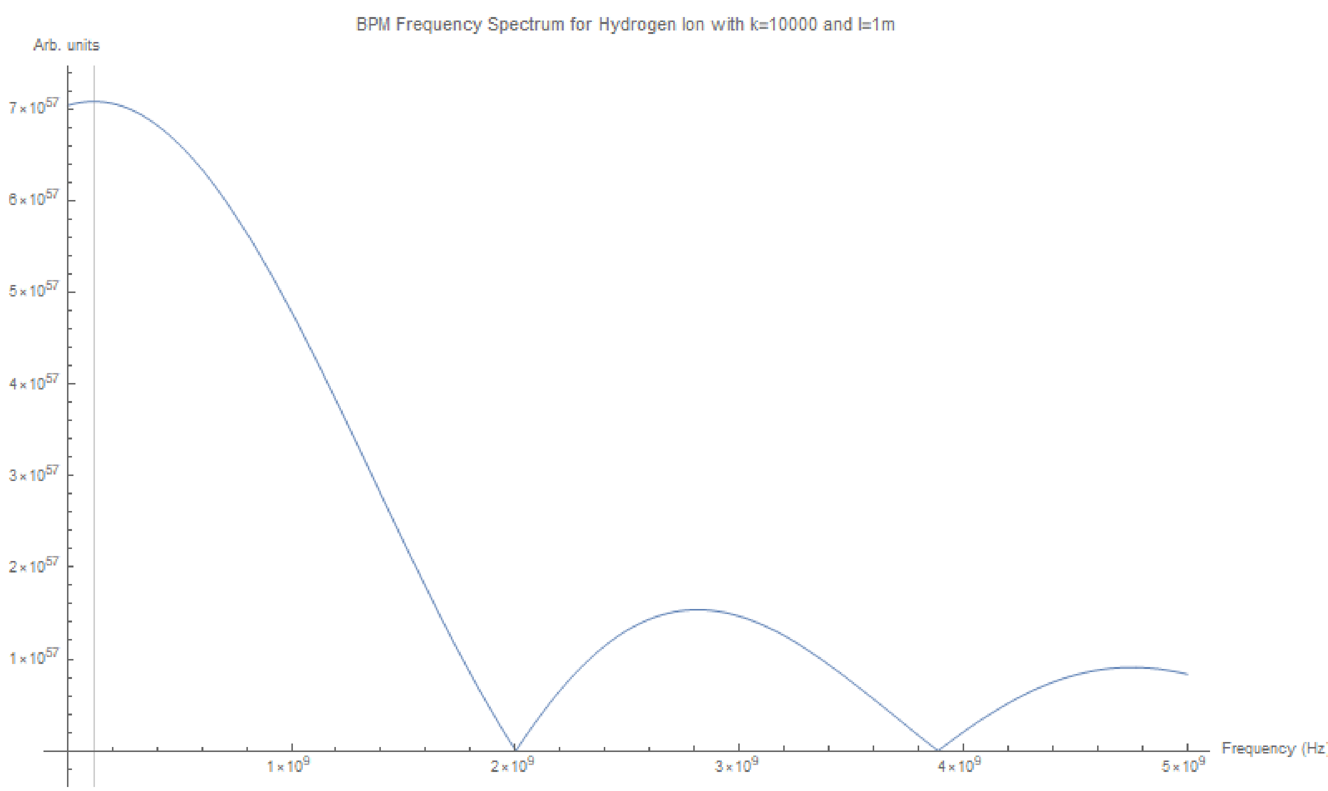

Figure 6: BPM Frequency Spectrum 6: H_2^+ with $k = 10000$ and $l = 1$m

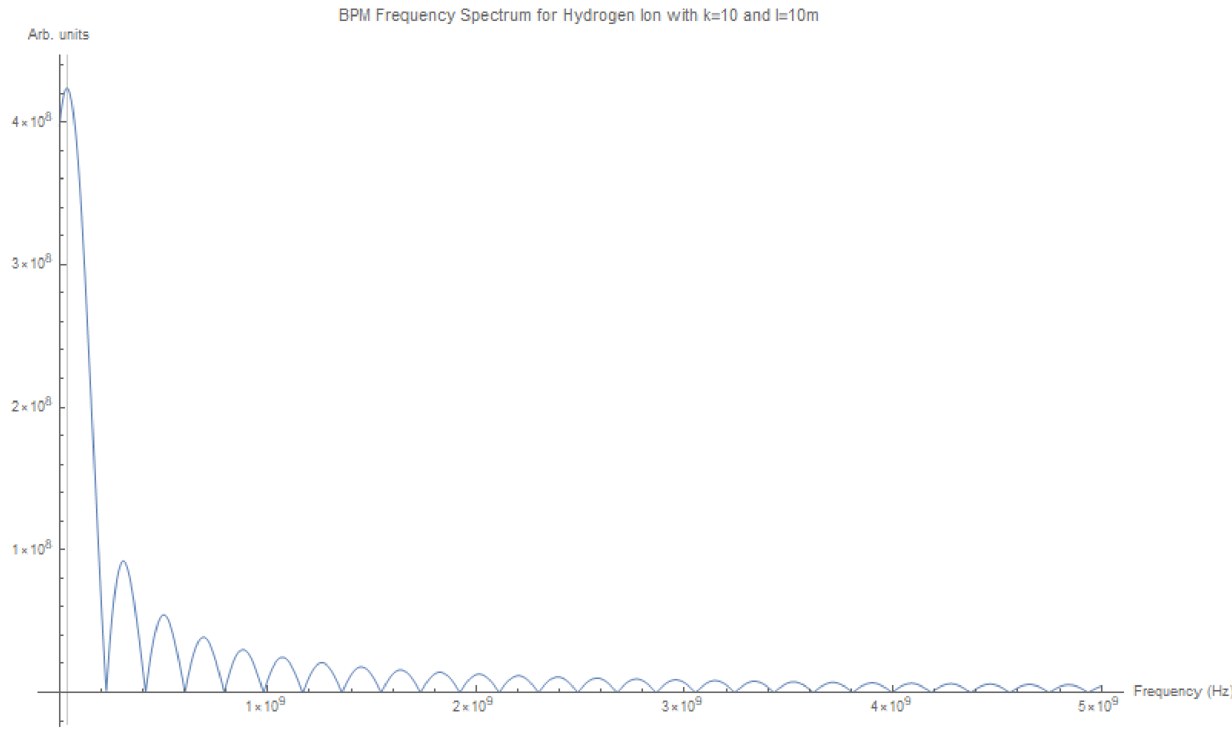

Figure 7: BPM Frequency Spectrum 7: H_2^+ with $k = 10$ and $l = 10$m

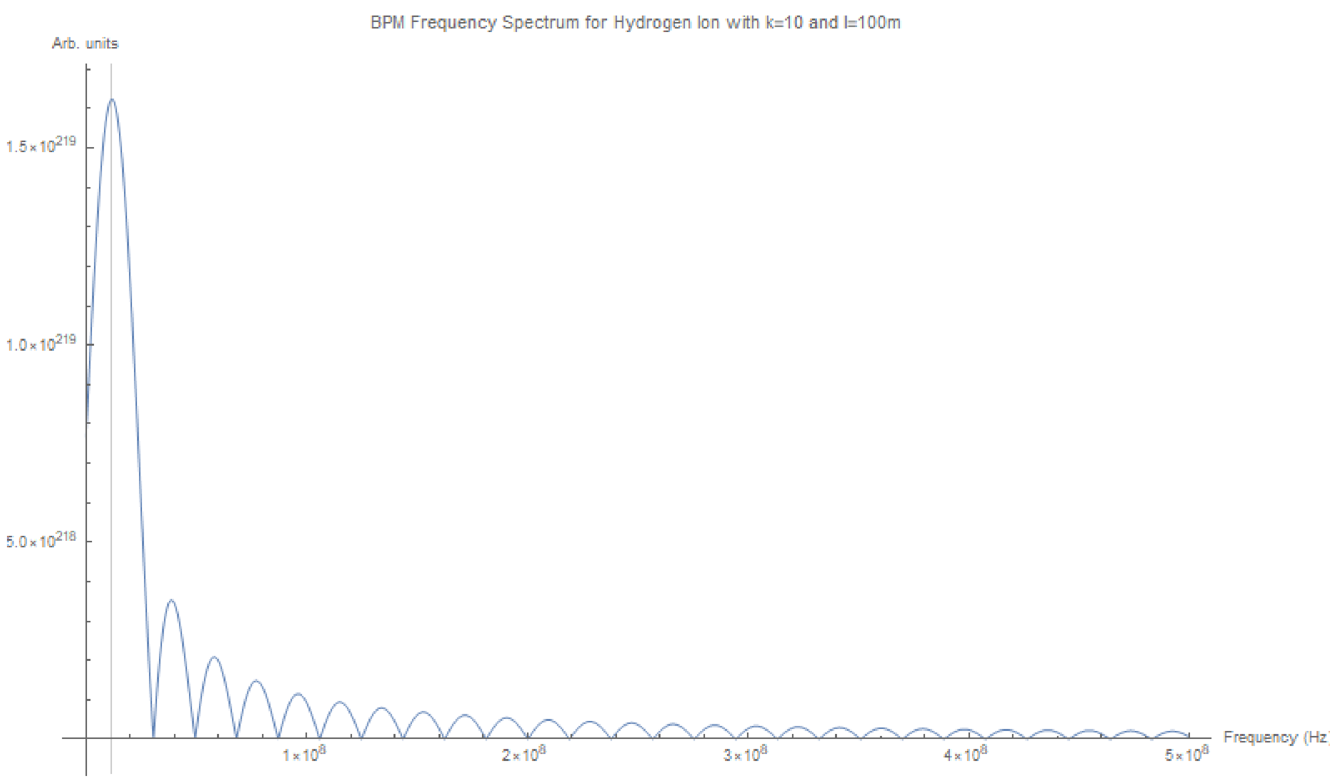

Figure 8: BPM Frequency Spectrum 8: H_2^+ with $k = 10$, $l = 100$m

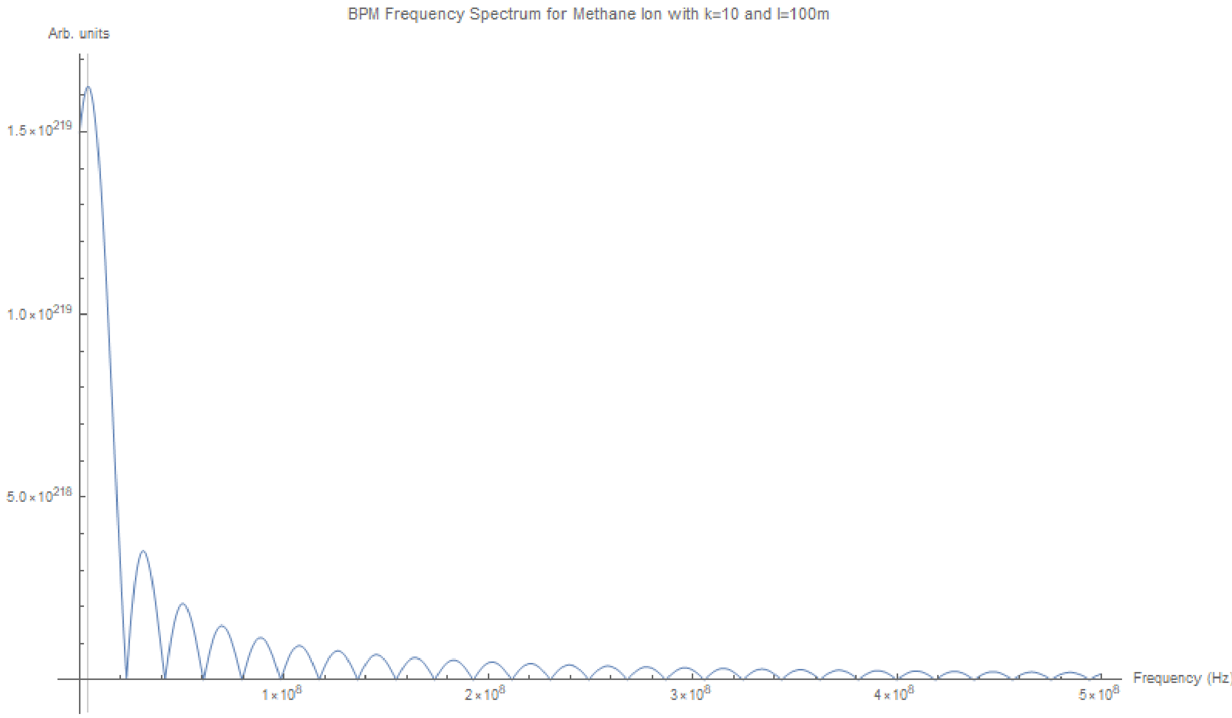

Figure 9: BPM Frequency Spectrum 9: CH_4^+ with $k = 10$ and $l = 100$m

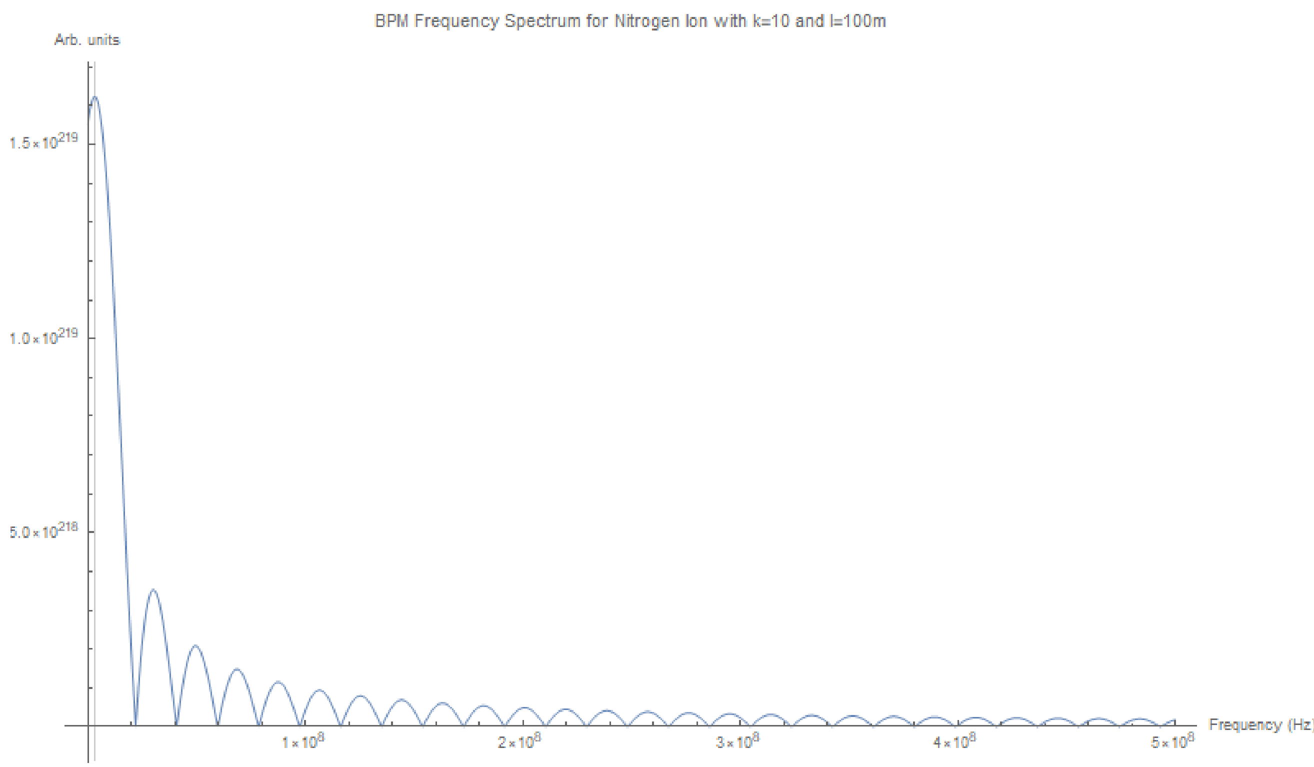

Figure 10: BPM Frequency Spectrum 10: N_2^+ with $k = 10$ and $l = 100$m

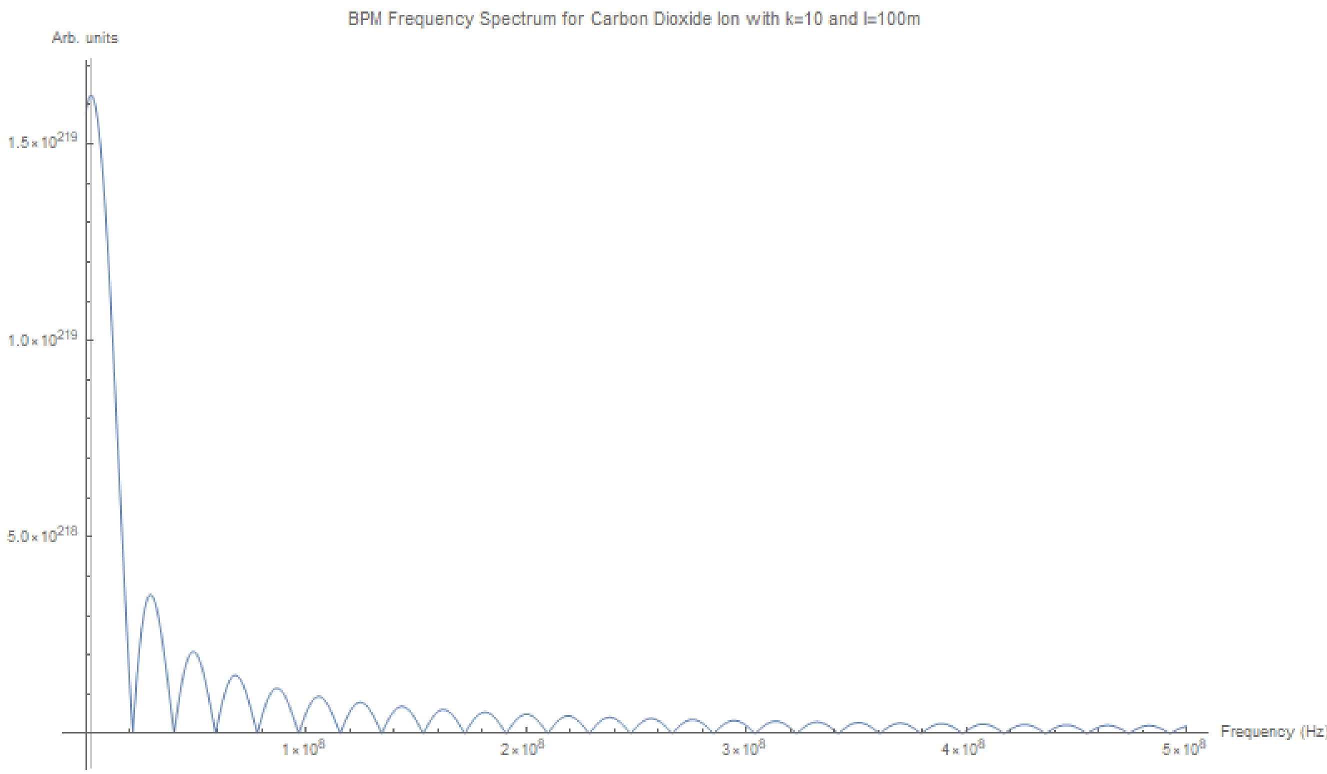

Figure 11: BPM Frequency Spectrum 11: CO_2^+ with $k = 10$ and $l = 100$m

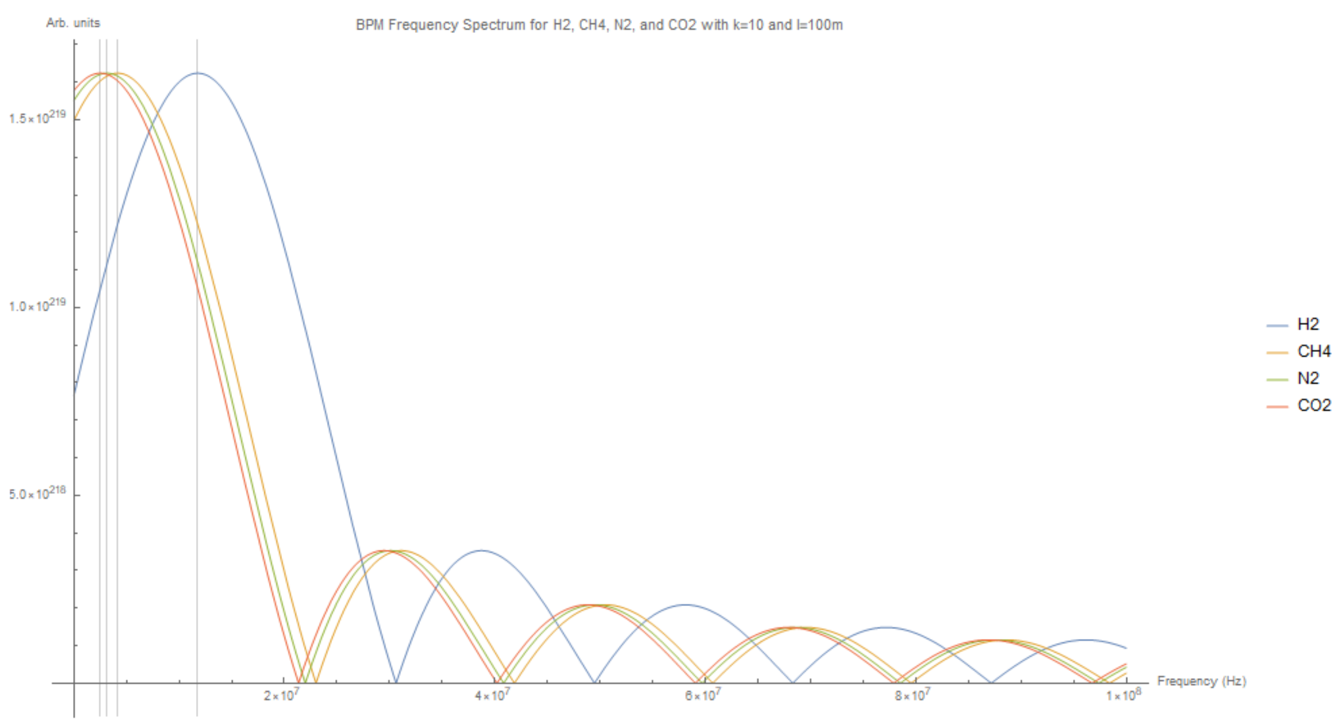

Figure 12: BPM Frequency Spectrum 12: H_2^+, CH_4^+, N_2^+ and CO_2^+ with $k = 10$ and $l = 100$m