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Laser beams with phase singularities

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Phase singularities in an optical field appear as isolated dark spots and can be generated in active laser cavities or by computer generated holograms. Detection and categorization of these singularities can easily be achieved either by interferometry or Fourier transform pattern recognition using a computer generated hologram.

1. Introduction

Most laser beams have essentially spherical wavefronts and any deviation from sphericity constitutes a degradation of beam quality. However, the TEM_{01}^* 'doughnut' mode, often observed in high-power lasers, can exhibit a *helical* wavefront structure, associated with a phase singularity on the beam axis [1]. This occurs when the frequency-degenerate TEM_{01} and TEM_{10} modes oscillate simultaneously in phase quadrature.

The electric field will have the form

$$\begin{aligned} E_{01}^* &= E_0[(x \pm iy)/\omega]\, e^{-(x^2+y^2)/\omega^2}\, e^{-ikr^2/2R}\, e^{i(kz+\Phi)} \\ &= E_0(r/\omega)\, e^{\pm i\theta}\, e^{-(x^2+y^2)/\omega^2}\, e^{-ikr^2/2R}\, e^{i(kz+\Phi)} \end{aligned} \tag{1}$$

where Φ is the Guoy phase shift, R is the wavefront radius of curvature, ω is the spot size and r and θ are polar coordinates in the X–Y plane. Notice that during any complete circuit around the axis the phase changes by $\pm 2\pi$, expressing the helical form of the wavefronts, and that the field goes to zero on the axis, as it must where the phase is undefined.

The doughnut mode is a stable cavity mode because it is a linear combination of Hermite–Gaussian TEM_{01} and TEM_{10} modes and propagates in a self-similar way in free space and through optical systems for the same reason. In fact other more complex combinations containing phase singularities can be produced from higher-order modes. According to resonator theory [2] Hermite–Gaussian modes TEM_{mn} with the same total $m + n$ (or Laguerre–Gaussian modes TEM_{pl} with the same total $2p + l$) are frequency degenerate and can form such combinations. The singularities show up as multiple isolated irradiance zeros in the modal spot pattern.

One example is the TEM_{02}^* hybrid or 'optical leopard' [3] which can be constructed from the Gaussian–Hermite modes TEM_{02} and TEM_{20}. The pattern has a central irradiance peak surrounded by four smaller peaks and four zeros where there are two positive and two negative singularities diagonally opposed. Figure 1a shows the irradiance distribution and Fig. 1b the form of the wavefronts with four interconnected helices.

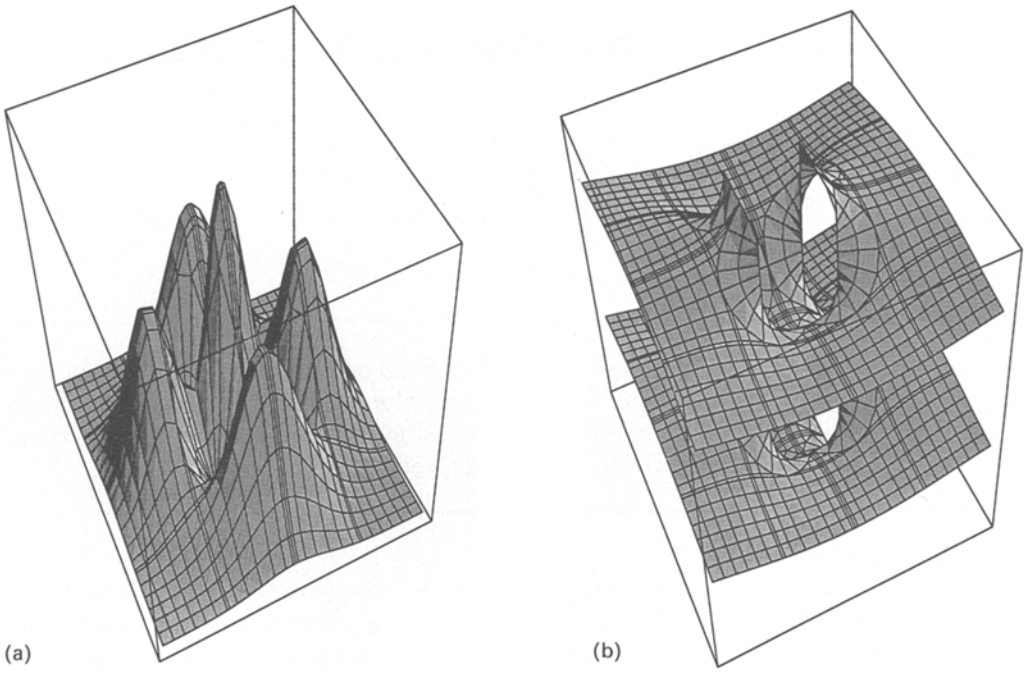


Figure 1 TEM_{02}^{*} or 'optical leopard'. (a) Irradiance distribution (vertical) with spatial position (horizontal). (b) Form of wavefronts. In the diagram, phase increases in the vertical direction. Two surfaces of constant phase differing by 2π, corresponding to a plane wave at large distances from the beam axis, are connected around the singularities.

Other examples contain higher-order singularities. The simplest cases are the higher-order doughnuts (which are just Gaussian–Laguerre TEM_{0n} modes)

$$E_{0n}^{*} = E_0(r/\omega)^n \, e^{-(x^2+y^2)/\omega^2} \, e^{\pm in\theta} \, e^{-ikr^2/2R} \, e^{i(kz+\Phi)} \tag{2}$$

The parameter n is often referred to as the 'charge' of the singularity, with zero charge corresponding to a Gaussian TEM_{00} beam. The irradiance profiles of several higher-order doughnuts are shown in Fig. 2. The wavefronts form multistart helices.

Of course, in a real laser, astigmatism in the cavity often removes the frequency degeneracy between such modes, but Brambilla *et al.* [3] have shown theoretically and experimentally that, so long as the astigmatism is not too severe, a cooperative frequency-locking process can occur, leading to a range of stable patterns.

However, it is important to remember that not all dark spots in patterns are necessarily phase singularities. If the frequency degeneracy of the contributing modes is broken, a rapidly time-varying pattern will result, the time average of which may still contain dark spots [1]. In some applications, involving only average irradiances, that will not matter, but in others it will be important. We show below how such cases can be distinguished experimentally.

2. Production of beams with singularities

There are several ways in which beams with singularities might be produced. As mentioned above, thanks to the process of cooperative frequency locking [3] such modes can arise

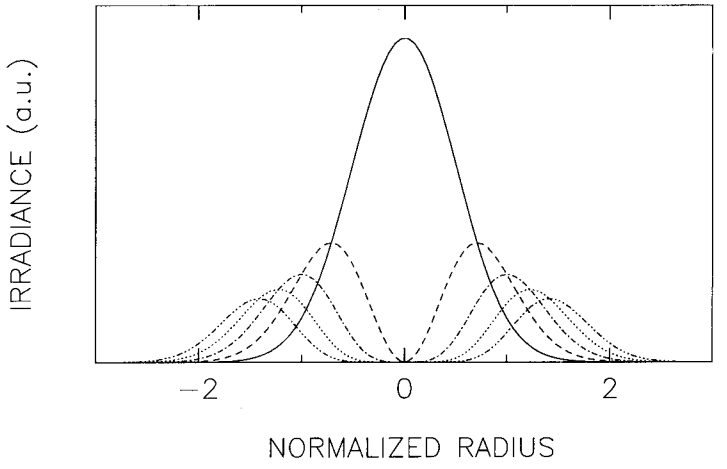


Figure 2 Irradiance profiles for Gaussian and doughnut modes up to charge 4. Each has the same total power.

spontaneously within a laser and in some cases a considerable degree of control can be exercised by the experimenter. Alternatively, a hologram can be used to convert part of the output of an existing laser into the desired beam. We consider both approaches below.

The 525 nm Na_2 vapour laser pumped by the Ar-ion laser [4] is ideal for studies of combination-mode formation because its velocity-selective optical pumping mechanism leads to very narrow gain linewidths so that only one 'family' of transverse modes with a certain $m + n$ or $2p + l$ sum can oscillate at a time, and the otherwise dominant TEM_{00} Gaussian can be suppressed. We have used a system very similar to that used by Brambilla *et al.* [3] to generate a range of patterns and, as described below, to study methods to detect and classify modes with singularities.

A less efficient but a more flexible way to produce modes with singularities is through the use of computer-generated holograms. In our first experiments we used on-axis holograms which have the form of spiral Fresnel zone plates [5], but these suffer the same problem as Gabor's original holograms, i.e. lack of separation between reconstructed beam and incident beam. It turns out that even crude off-axis binary holograms are quite effective, as will now be explained.

A hologram is really just a recording of the interference pattern between a field of interest and some simple reference field. For the relatively simple fields involved in modes with singularities it is possible to calculate the form of such patterns and plot them out. Let us take as an example a charge-one doughnut (Equation 1), at a beam-waist ($R \to \infty$) for simplicity. Consider the interference pattern on a screen in the X–Y plane when a plane reference beam

$$R = R_0 \, e^{ik_x x + ik_z z} \tag{3}$$

is incident at an angle $\phi = \sin^{-1}(k_x/k)$. The irradiance on a screen at $z = 0$ will be

$$\begin{aligned} I &= |(R_0 \, e^{ik_x x} + E_0(r/\omega) \, e^{i\theta} \, e^{-r^2/\omega^2}|^2 \\ &= R_0^2 + E_0^2(r/\omega)^2 \, e^{-2r^2/\omega^2} + 2R_0 E_0(r/\omega) \, e^{-r^2/\omega^2} \cos(k_x x - \theta) \end{aligned} \tag{4}$$

It is the last term which expresses the interference pattern. A photographic recording of this

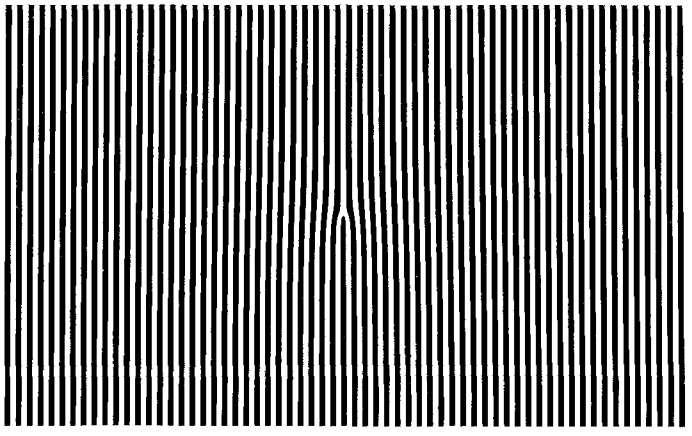


Figure 3 Pattern for off-axis binary hologram for charge-one singularity. The origin is at the tip of the fringe defect.

pattern can now act as a hologram capable of 'reconstructing' the original doughnut (and its complex conjugate) when illuminated by a wave given by Equation 3.

In what follows it will be convenient to work with a simplified pattern which ignores the amplitude variation of the doughnut beam and retains only the important phase information in the form of a spatially varying transmissivity

$$T = \tfrac{1}{2}(1 - \cos(k_x x - \theta)) \tag{5}$$

Now consider the effect of illuminating this pattern with a Gaussian beam propagating along the axis. Just after the hologram the field will be

$$E_T = TA_0 \, \mathrm{e}^{-r^2/\Omega_0^2} \tag{6}$$

where A_0 is the central amplitude and Ω_0 is the spot size of the beam, assumed plane at this point. Substituting for T, we find

$$E_T = (A_0/2)\,\mathrm{e}^{-r^2/\Omega_0^2} - (A_0/4)\,\mathrm{e}^{-r^2/\Omega_0^2}\,\mathrm{e}^{\mathrm{i}(k_x x - \theta)} - (A_0/4)\,\mathrm{e}^{-r^2/\Omega_0^2}\,\mathrm{e}^{\mathrm{i}(-k_x x + \theta)} \tag{7}$$

This field can be recognized as consisting of a zero-order beam propagating along the axis, and two (conjugate) first-order diffracted beams, each of them containing a singularity of opposite charge.

In fact it is much easier to print binary holograms than ones incorporating the sinusoidal variations in optical density implicit in Equation 5. Thus we actually use a 'square wave' transmissivity function which can be expressed in the form

$$T = \tfrac{1}{2} - \sum_{n=1}^{\infty} \operatorname{sinc}(n\pi/2) \cos[n(k_x x - \theta)] \tag{8}$$

Its appearance is shown in Fig. 3. It has the appearance of a grating with a defect where a stripe branches.

When this is illuminated, the output field will contain terms of the form

$$E_n = (A_0/4) \operatorname{sinc}(n\pi/2)\,\mathrm{e}^{-r^2/\Omega_0^2}\,\mathrm{e}^{\mathrm{i}(nk_x x - n\theta)} \tag{9}$$

Each is of course a diffraction order from the 'grating' and can be recognized as being

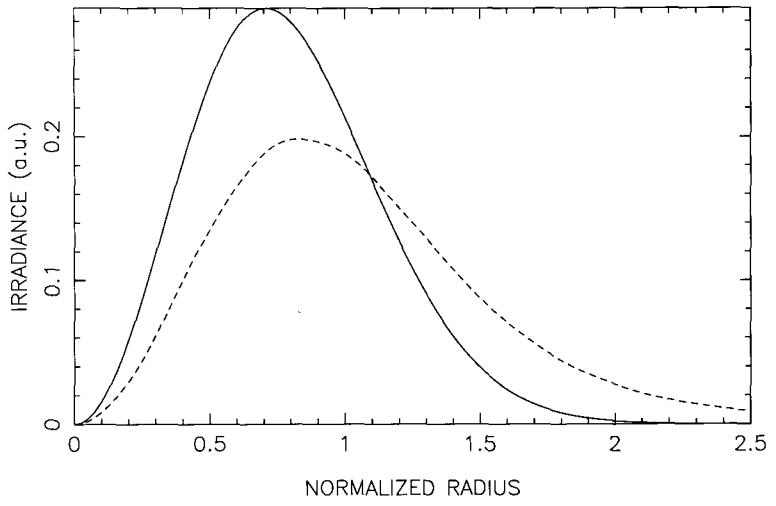


Figure 4 Comparison of the far-field spatial profile of the TEM_{01}^* mode (full line) with a first-order beam from a hologram illuminated by a Gaussian beam with the same parameters (dashed line). The TEM_{01}^* mode has the same power as the zeroth term of the series in Equation 13.

closely related to an nth-order doughnut propagating at an angle

$$\phi_n = \sin^{-1}(nk_x/k) \simeq n \sin^{-1}(k_x/k) \tag{10}$$

to the axis. Actually, Equation 9 has the form of a charge-n singularity embedded in a Gaussian beam and as such will not propagate in a self-similar way. To investigate the far-field spatial profile of this beam, we can decompose Equation 9 in terms of the orthogonal set of Gaussian–Laguerre modes:

$$\begin{aligned}\psi_{pl} = {} & \sqrt{\left(\frac{2p!}{\pi(p+l)!}\right)}\left(\frac{\Omega_0}{\Omega_1}\right)\left(\frac{\sqrt{2}r}{\Omega_1}\right)^{|l|} L_p^{|l|}\left(\frac{2r^2}{\Omega_1^2}\right) \\ & \times \mathrm{e}^{-r^2/\Omega_1^2}\, \mathrm{e}^{\mathrm{i}l\theta}\, \mathrm{e}^{-\mathrm{i}kr^2/2R}\, \mathrm{e}^{\mathrm{i}(2p+l+1)\Phi_1}\, \mathrm{e}^{\mathrm{i}(nk_x x + k_z z)}\end{aligned} \tag{11}$$

where Ω_1 is the spot size, R_1 is the radius of curvature of the beam and Φ_1 is the Guoy phase shift. As the beam propagates, the parameters Ω_1, R_1 and Φ_1 are related to the waist spot size Ω_0 via the standard Gaussian–Laguerre propagation laws [6]. The amplitudes of the terms of the series expressing the field at the hologram are given by

$$\begin{aligned} E_{pl} &= 0 \quad \text{if} \quad l \neq n \\ E_{pn} &= \frac{A_0}{4}\,\mathrm{sinc}\,(n\pi/2)\Omega_0^2 \sqrt{\left(\frac{\pi p!}{2(p+n)!}\right)}\,\frac{n}{2}\,\frac{\Gamma(p+n/2)}{p!}\end{aligned} \tag{12}$$

In the far-field the spatial amplitude distribution is

$$E_{\text{far}} = \sum_{p=0}^{\infty} E_{pn}\psi_{pn} \tag{13}$$

where $\Phi_1 = \pi/2$. Figure 4 shows that the far-field amplitude distribution for the first-order beam closely resembles a TEM_{01}^* doughnut of slightly increased spot size. The helical structure of the wavefronts is identical. Similar results are obtained for higher orders. Thus,

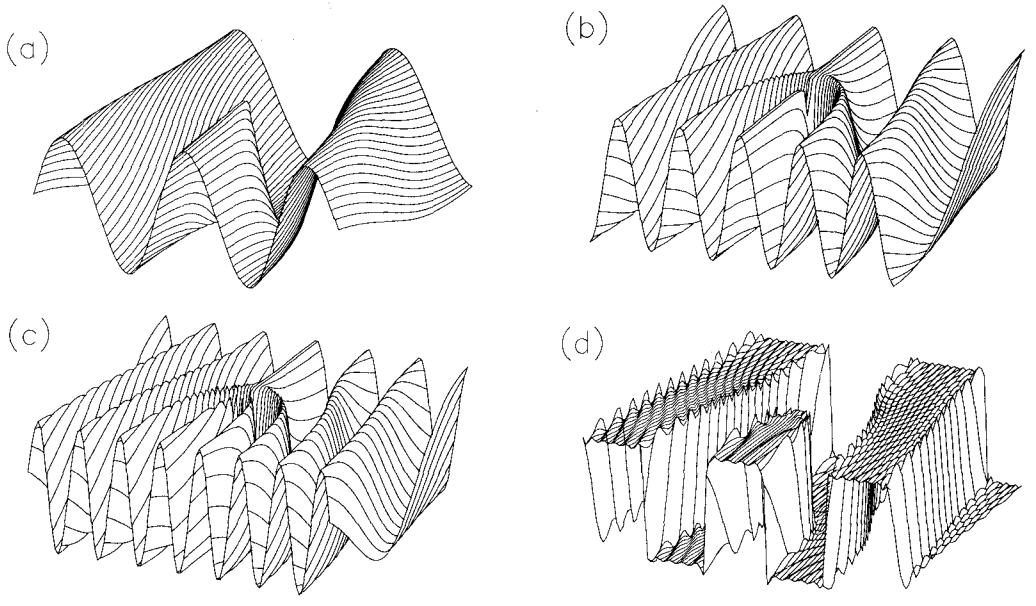
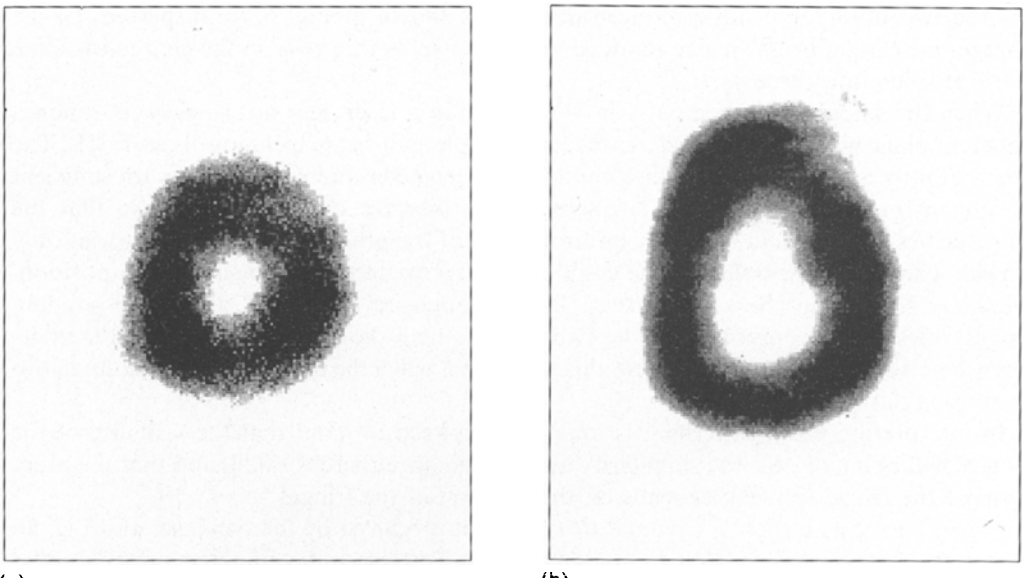


Figure 5 Decomposition of binary hologram pattern into harmonic components: (a–c) first, second and third components; (d) synthesis of first 30 components.

although the beams produced by our holograms are not strictly TEM^*_{0n} doughnuts, in the far-field they are, for all practical purposes, equivalent and for simplicity we refer to them as such.

The way in which the binary pattern produces multiply charged singularities can be understood by reference to Fig. 5, where the first three harmonic components of the very centre of the pattern in Fig. 3 are plotted. Each of Figs 5a to 5c can be recognized as the hologram for a successively higher-order singularity. Figure 5d shows the square-wave pattern synthesized from 30 components.

Figure 3 showed an example of a computed binary off-axis hologram pattern. This was printed by a laser printer on A4 paper and reduced by photographing onto half of a 35-mm slide. Figures 6a and 6b show charge-one and charge-two doughnuts produced as first and second orders, using a HeNe laser for illumination. The patterns, like all subsequent experimental ones, were recorded by an Electrim EDC-1000 CCD camera. About 5% of the incident power was coupled into the first order. This could be improved considerably by converting the present hologram into a phase hologram if facilities were available. Although the charge-two pattern is somewhat distorted, it shows clearly the larger diameter and narrower bright annulus expected. In fact, a symmetric grating like ours would be expected to produce a dim second-order term at best as it should effectively be a 'missing order'. We have also used holograms with four fringe defects to generate an 'optical leopard' pattern and still more complex beam shapes could be produced. It would even be possible to generate combinations involving transverse modes of different $m + n$ (or $2p + l$) sum (e.g. $TEM_{00} + TEM^*_{01}$) which could not be generated at a single frequency in a laser oscillator. These would suffer some change of shape during propagation owing to differential Guoy phase shift, but perhaps that could be put to some use.

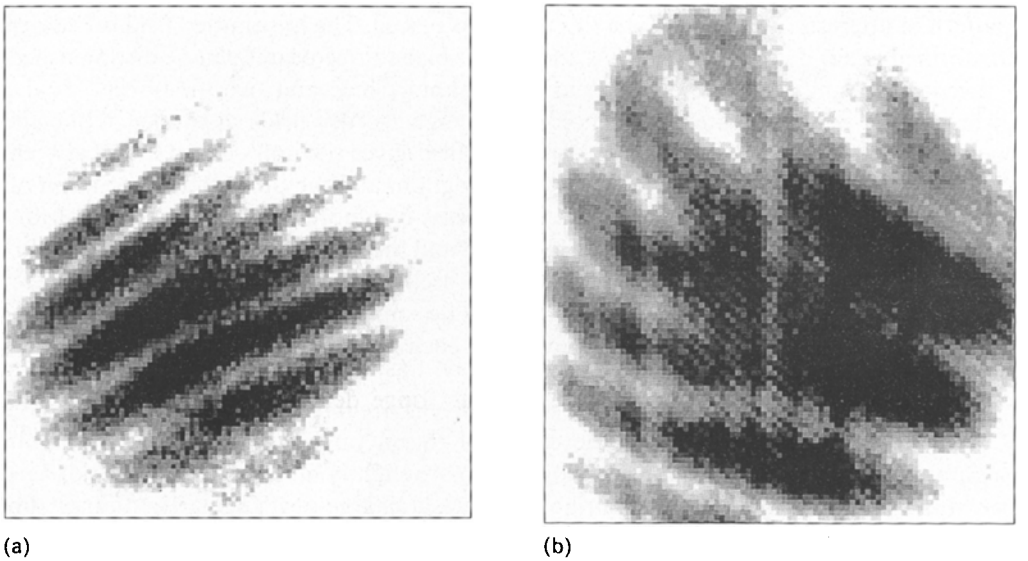


(a) (b)

Figure 6 Doughnut beams produced by hologram using HeNe laser light: (a) charge one; (b) charge two.

3. Detection of phase singularity modes

In their studies of doughnut modes, Tamm and Weiss [7] used an astigmatic imaging technique to determine the charges of the singularities, but a more versatile approach is to measure the phase structure of the beam by interference, as had been done by Vaughan and Willetts [1]. Ideally, one should have available a coherent plane wave to act as reference, and in the vicinity of each singularity a fringe pattern with a defect, like that in Fig. 3, would be observed. Exactly this behaviour is demonstrated in Fig. 7, where interference patterns



(a) (b)

Figure 7 Interference of doughnuts of Fig. 6 with plane wave: (a) charge one; (b) charge two.

for the two holographically produced doughnuts shown in Fig. 6 are displayed. In the charge-one case, a bright fringe (dark in this print) splits into two; in the charge-two case a fringe splits into three.

When the singularity mode pattern is produced in a laser, it is not so easy to obtain a coherent plane wave for use as reference, and a simple split-beam technique is useful [1]. The beam of interest is split in a Mach–Zehnder interferometer and recombined with sufficient misalignment to produce straight fringes and to displace the two patterns so that the singularities in one fall in relatively uniform areas of the other where the phase varies only slowly. The result is a defect in the combined fringe pattern at each singularity position. Opposite charges in the same pattern fork in opposite directions, and corresponding singularities behave oppositely at the two points where they appear. Although the interference pattern becomes complicated, this works well when the pattern does not contain too many singularities [8].

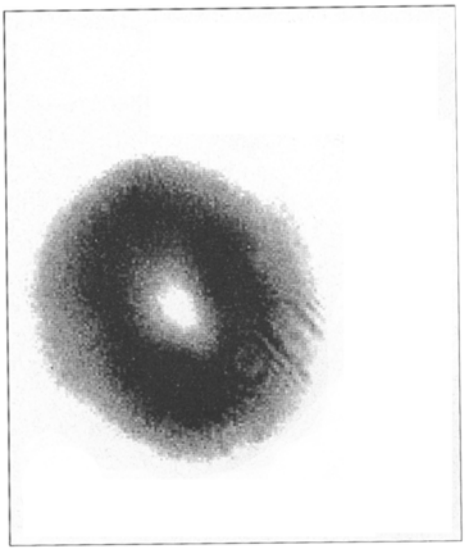
In interpreting these patterns it is important to keep in mind that the visibility of the fringes will be small near the singularity as the irradiance there is small, and that the exact form of the fringe splitting depends on the position of the fringes.

Figure 8a shows a slightly asymmetric doughnut produced by the Na_2 laser and Fig. 8b shows the corresponding split-beam interference pattern, indicating the presence of a charge-one phase singularity in the beam. For comparison Fig. 8c shows a split-beam interference pattern for an 'unlocked' doughnut, i.e. one where cavity astigmatism has broken the frequency degeneracy of the constituent modes. This shows the characteristic 'sideways shift' of the fringes inside a circular region as explained by Vaughan and Willetts [1]. The loss of frequency degeneracy leads to mode beating at a frequency of a few megahertz which can be detected with a photodiode.

An alternative means of detecting and classifying phase singularities in a beam is by optical Fourier transform recognition techniques where the holograms discussed above can be used as matched filters [9].

This has been demonstrated using the arrangement shown in Fig. 9. The Fourier transform of an input pattern is formed at the focal plane of a lens where the hologram for a pattern of interest, say a charge-one singularity, is placed. The transmitted field is Fourier transformed again by a second lens. At the output plane three beams can be distinguished – a central magnified image of the field at the input plane, and two 'first-order' fields. (Owing to the binary nature of our holograms, higher-order fields also appear but are generally too weak to be useful.) One first-order field gives the cross-correlation between the input field and the field used to make the hologram, and the other the convolution of the input field with the hologram field [9]. In this case it is more helpful to realize that this is also the cross-correlation between the input field and the conjugate of the hologram field. Thus, if a hologram for a charge-one singularity is used as a filter, a bright spot will appear in one field at each point where a positive charge-one singularity appears in the input field, and in the other field a bright spot will appear at each point where a negative charge-one singularity appears in the input.

This is shown in Fig. 10a where the singularity in a charge-one doughnut produced by the Na_2 laser is recognized. At the centre of the picture is the image of the input doughnut, flanked by the recognition fields (which are slightly magnified as a result of lens aberrations). The intense spot on the right indicates a charge-one singularity. Figure 10b shows the result when the input field is a 'leopard'. Here the diagonal placement of the four charge-one singularities, two of each sign, shows up clearly. An unlocked doughnut gives



(a)

(b)

(c)

Figure 8 Doughnuts produced by Na_2 laser. (a) Phase-locked doughnut. (b) Split-beam interference pattern for (a). Note two forks in the fringe pattern, indicating the presence of a singularity. (c) Split-beam interference pattern for unlocked doughnut.

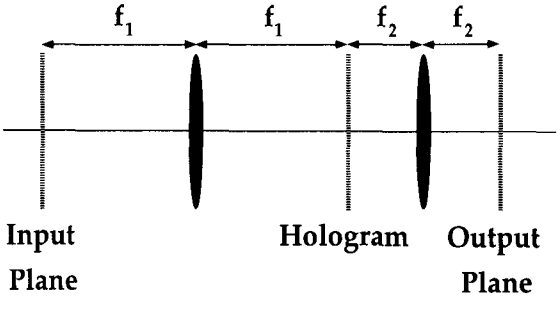


Figure 9 Arrangement for Fourier transform recognition of singularities in beams.

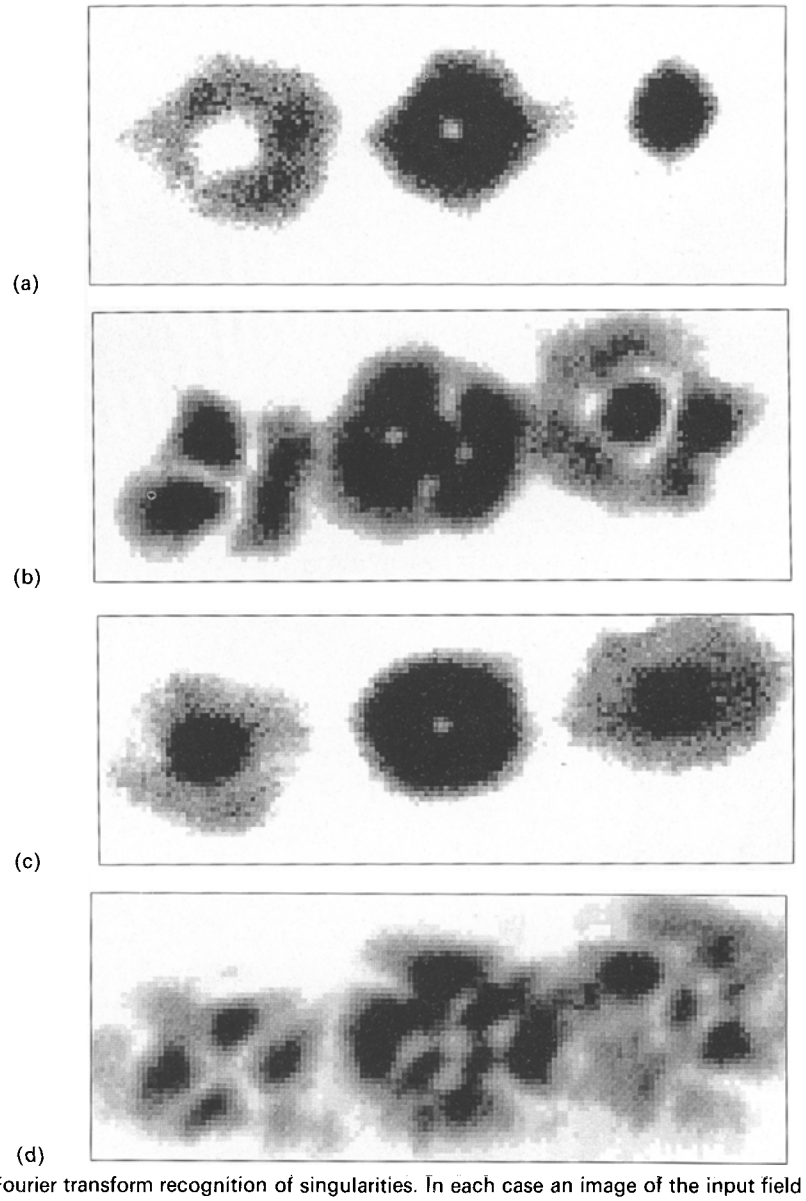


Figure 10 Fourier transform recognition of singularities. In each case an image of the input field appears at the centre, flanked by recognition fields for ±1 charged singularities: (a) Charge-one doughnut from the Na_2 laser. (b) 'Leopard' from the Na_2 laser. (c) Unlocked $m + n = 1$ doughnut from Na_2 laser. (d) Seven-spot beam from Na_2 laser.

the result shown in Fig. 10c, as the actual pattern is rapidly changing, spending part of each beat period as a doughnut of each sign [1], so that bright spots appear in the recording on both sides. A more complex pattern from the Na_2 laser, with seven dark spots, is analysed in Fig. 10d, which shows that four charge-one singularities of one sign form a square with the three others of opposite sign lying along a line. Such a pattern, called a 'seven-hole' by Brambilla *et al.* [3], would be difficult to analyse using split-beam interference.

Although there is clearly room for improvement, the optical Fourier recognition technique has the advantage of needing only a single beam and providing output in the form of readily recognized spots rather than 'forks' in an interference pattern. Indeed, it is interesting to note that, presumably owing to the local rotational symmetry and self similarity of the regions surrounding singularities, the technique is less affected by variations in scale-size or orientation than is, say, recognition of alphanumeric characters.

4. Applications

This work arose originally out of an interest in pattern formation and optical turbulence but it is clear that a 'technology' now exists for the production, detection, and classification of beams with phase singularities associated with isolated irradiance zeros. Many lasers can be run on the TEM_{01}^{*} mode, and well-designed phase holograms could convert the light from almost any laser into such a form, or a more complex pattern [10].

In some cases the helical phase structure is of importance, as in switching helicities as a means of information processing [7], but in many other cases any beam with a central minimum in irradiance (thus including unlocked doughnuts) could be of use. Note that the structure we have been discussing is entirely independent of polarization. Such a shape could be advantageous when it is desired to launch a beam through a reflecting telescope with a central obstruction. A higher-order doughnut would be appropriate for this.

Another area of potential application is in light–particle interactions [11]. The use of doughnut beams has been suggested for small-particle levitation [12] and more recently for the focusing of atomic beams [13].

Finally, it should be pointed out that speckle patterns formed by laser beams passing through inhomogeneous media have been shown to contain many phase singularities [14] presumably produced by random structures approximating the holograms discussed above.

5. Conclusions

We have shown that laser beams with phase singularities can be generated in lasers or can be produced from normal Gaussian beams using computer-generated holograms. The presence of the singularities can be detected and they can be classified either by reference to defects in interference patterns or by optical Fourier transform pattern-recognition techniques using computer-generated holograms as matched filters. Possible applications of the simplest of such beams, the TEM_{01}^{*} doughnut, include efficient launching of single-mode beams through telescopes and atom and particle trapping.

Acknowledgement

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References

1. J. H. VAUGHAN and D. V. WILLETTS, *J. Opt. Soc. Am.* **73** (1983) 1018.
2. H. KOGELNIK and T. LI, *Proc. IEEE* **54** (1966) 1312.
3. M. BRAMBILLA, F. BATTIPEDE, L. A. LUGIATO, V. PENNA, F. FRATI, C. TAMM and C. O. WEISS, *Phys. Rev.* **A43** (1991) 5090.
4. B. WELLEGEHAUSEN, *IEEE J. Quantum Electron.* **QE-15** (1979) 1108.
5. N. R. HECKENBERG, R. McDUFF, C. P. SMITH and A. G. WHITE, *Opt. Lett.* **17** (1992).
6. J. P. TACHÉ, *Appl. Opt.* **26** (1987) 2698.
7. CHR. TAMM and C. O. WEISS, *J. Opt. Soc. Am.* **B7** (1990) 1034.
8. A. G. WHITE, C. P. SMITH, N. R. HECKENBERG, H. RUBINSZTEIN-DUNLOP, R. McDUFF, C. O. WEISS and CHR. TAMM, *J. Mod. Opt.* **38** (1991).

9. J. W. GOODMAN, *Introduction to Fourier Optics* (McGraw-Hill, San Francisco, 1968).
10. Phase holograms for charge-one singularities have already been produced by P. Vanetti, Alenia Sistemi Difesi.
11. S. CHU, *Science* **253** (1991) 253.
12. A. ASHKIN and J. M. DZIEDZIC, *Appl. Phys. Lett.* **24** (1974) 586.
13. J. J. McCELLAND and M. R. SCHEINFEIN, *J. Opt. Soc. Am.* **B8** (1991) 1974.
14. N. B. BARANOVA, B. YA. ZEL'DOVICH, A. V. MAMAEV, N. F. PILIPETSKIL and V. V. SHKUKOV, *JETP Lett.* **33** (1981) 195.