# Error Analysis

04 September 2013

# Penfold-Leiss Cross Section Unfolding

- Measure Yields at: $E = E_1, E_2, \ldots, E_n$ where,

$$E_i - E_{i-1} = \Delta, i = 2, n$$

$$Y(E_i) = \int_{th}^{E_i} n_\gamma(E_i, k)\sigma(k)dk \approx \sum_{j=1}^{i} N_\gamma(E_i, \Delta, E_j)\sigma(E_j)$$

- The solution can be written in two forms:

$$\sigma_i = \frac{1}{N_{ii}}\left[y_i - \sum_{j=1}^{i-1}\left(N_{ij}\sigma_j\right)\right]$$

- Or, Matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} N_{\gamma,11} & 0 & \cdots & 0 \\ N_{\gamma,21} & N_{\gamma,22} & \cdots & 0 \\ \ddots & \ddots & \ddots & 0 \\ N_{\gamma,n1} & N_{\gamma,n2} & \cdots & N_{\gamma,nn} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix}$$

$$[Y] = [N] \bullet [\sigma]$$

$$[\sigma] = [N]^{-1} \bullet [Y]$$

# Statistical Error

# Statistical Error Propagation (1)

- Note: $\frac{dy_i}{y_i} = \frac{1}{\sqrt{y_i}}$ $\frac{dN_{ij}}{dN_{ij}} = \frac{1}{\sqrt{N_{ij}}} \approx 0$

$$dy_i = \sqrt{y_i} \qquad dy_i = \sqrt{y_i + 2y_i^{bg}}$$

In case of background Subtraction

- With:

$$[B] = [N]^{-1}$$

$$[\sigma] = [B] \bullet [Y]$$

- Then:

$$[d\sigma^2] = [B] \bullet [dY^2] \bullet [B]^T$$

- Where:

$$\left[dY^2\right] = \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & y_n \end{bmatrix}$$

$$\mathrm{var}(y_i, y_i) = y_i$$
$$\mathrm{cov}(y_i, y_j) = 0$$

$$\left[d\sigma^2\right] = \begin{bmatrix} d\sigma_1^2 & \mathrm{cov}(\sigma_1, \sigma_2) & \cdots & \mathrm{cov}(\sigma_1, \sigma_n) \\ \mathrm{cov}(\sigma_2, \sigma_1) & d\sigma_2^2 & \cdots & \vdots \\ \ddots & \ddots & \ddots & \vdots \\ \mathrm{cov}(\sigma_n, \sigma_1) & \cdots & \cdots & d\sigma_n^2 \end{bmatrix}$$

# Statistical Error Propagation (2)

$$(d\sigma_i)^2 = \frac{1}{N_{ii}^2}\left[ dy_i^2 + \sum_{j=1}^{i-1} N_{ij}^2 d\sigma_j^2 + \sum_{k=1}^{i-1}\sum_{l=1}^{i-1} N_{ik}\,\mathrm{cov}(\sigma_k,\sigma_l) N_{il} \right]$$

Although,

$$\mathrm{cov}(y_i, y_j) = 0,$$

$$\mathrm{cov}(\sigma_i, \sigma_j) \neq 0$$

For mono-chromatic beam

$$\left(\frac{d\sigma_i}{\sigma_i}\right)^2 = \left(\frac{dy_i}{y_i}\right)^2 = \frac{1}{y_i}$$

# Statistical Error Propagation (Wrong)

$$[\sigma] = [N]^{-1} \bullet [Y]$$

- Then:

Wrong

$$\left[d\sigma^2\right] = \left[N^2\right]^{-1} \bullet \left[dY^2\right]$$

- This is equivalent to:

$$(d\sigma_i)^2 = \frac{1}{N_{ii}^2}\left[dy_i^2 - \sum_{j=1}^{i-1} N_{ij}^2 d\sigma_j^2\right]$$

Wrong

# Systematic Error

# Systematic Error Propagation (1)

- For absolute beam energy uncertainty of δE:

$$dy_i = y_i(E+\delta E) - y_i(E)$$

Measured

$$dN_{ij} = N_{ij}(E+\delta E) - N_{ij}(E)$$

Simulated

- With:

$$[B] = [N]^{-1}$$

$$[\sigma] = [B] \bullet [Y]$$

- Then:

$$[d\sigma^2] = [B] \bullet \left([dY^2] + [dN^2] \bullet [\sigma^2]\right) \bullet [B]^T$$

- Where:

$$\left[dN^2\right] = \begin{bmatrix} dN_{11}^2 & 0 & \cdots & 0 \\ dN_{21}^2 & dN_{22}^2 & \cdots & 0 \\ \ddots & \ddots & \ddots & 0 \\ dN_{n1}^2 & dN_{n1}^2 & \cdots & dN_{nn}^2 \end{bmatrix}$$

$$\left[\sigma^2\right] = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & \vdots \\ \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma_n^2 \end{bmatrix}$$

# Systematic Error Propagation (2)

$$(d\sigma_i)^2 = \frac{1}{N_{ii}^2}\left[ dy_i^2 + \sum_{j=1}^{i-1} N_{ij}^2 d\sigma_j^2 + \sum_{k=1}^{i-1}\sum_{l=1}^{i-1} N_{ik} \operatorname{cov}(\sigma_k, \sigma_l) N_{il} \right.$$

$$\left. + \sum_{j=1}^{i-1} dN_{ij}^2 \sigma_j^2 + dN_{ii}^2 \sigma_i^2 \right]$$

# Other Systematic Errors

| | |
|---|---|
| **Beam Current, δI/I** | **3%** |
| Photon Flux, δN/N | 5% |
| Radiator Thickness, δR/R | 3% |
| Bubble Chamber Efficiency, ε | 5% |
| Bubble Chamber Thickness, δT/T | 3% |

Simulation

- Then:

$$(dy_i)^2 = (dy_i(\delta E))^2 + \left[\left(\frac{\delta T}{T}\right)^2 + \varepsilon^2\right] y_i^2$$

$$(dN_{ij})^2 = (dN_{ij}(\delta E))^2 + \left[\left(\frac{\delta I}{I}\right)^2 + \left(\frac{\delta N}{N}\right)^2 + \left(\frac{\delta R}{R}\right)^2\right] N_{ij}^2$$

# Results

I. Radiator Thickness = 0.02 mm
II. Bubble Chamber Thickness = 3.0 cm
III. Background subtraction of $^{18}O(\gamma,\alpha)^{14}C$
IV. $^{17}O(\gamma,n)^{16}O$: Still to do

| Electron Beam K. E. | Beam Current (μA) | Time (hour) | $y_i$ | $dy_i$ (no bg) | $dy_i$ (with bg) |
|---|---|---|---|---|---|
| 7.9 | 100 | 100 | 545 | 23 | 134 |
| 8.0 | 100 | 20 | 581 | 24 | 77 |
| 8.1 | 80 | 10 | 852 | 29 | 60 |
| 8.2 | 20 | 10 | 634 | 25 | 40 |
| 8.3 | 10 | 10 | 812 | 28 | 39 |
| 8.4 | 4 | 10 | 746 | 27 | 36 |
| 8.5 | 2 | 10 | 763 | 28 | 32 |

$$[N] = \begin{bmatrix} 3.267e14 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9.782e13 & 6.439e13 & 0 & 0 & 0 & 0 & 0 \\ 5.013e13 & 3.858e13 & 2.539e13 & 0 & 0 & 0 & 0 \\ 1.494e13 & 1.236e13 & 9.514e12 & 6.258e12 & 0 & 0 & 0 \\ 8.540e12 & 7.369e12 & 6.097e12 & 4.692e12 & 3.086e12 & 0 & 0 \\ 3.801e12 & 3.370e12 & 2.908e12 & 2.406e12 & 1.852e12 & 1.217e12 & 0 \\ 2.075e12 & 1.875e12 & 1.663e12 & 1.435e12 & 1.187e12 & 9.137e11 & 6.004e11 \end{bmatrix}$$

| Electron Beam K. E. | Cross Section (nb) | Stat Error (no bg, %) | Stat Error (with bg, %) |
|---|---|---|---|
| 7.9 | 0.046 | 4.4 | 24.5 |
| 8.0 | 0.185 | 6.0 | 20.7 |
| 8.1 | 0.58 | 6.3 | 14.7 |
| 8.2 | 1.53 | 8.2 | 13.8 |
| 8.3 | 3.49 | 9.1 | 13.3 |
| 8.4 | 7.2 | 10.6 | 13.8 |
| 8.5 | 13.6 | 12.2 | 14.8 |

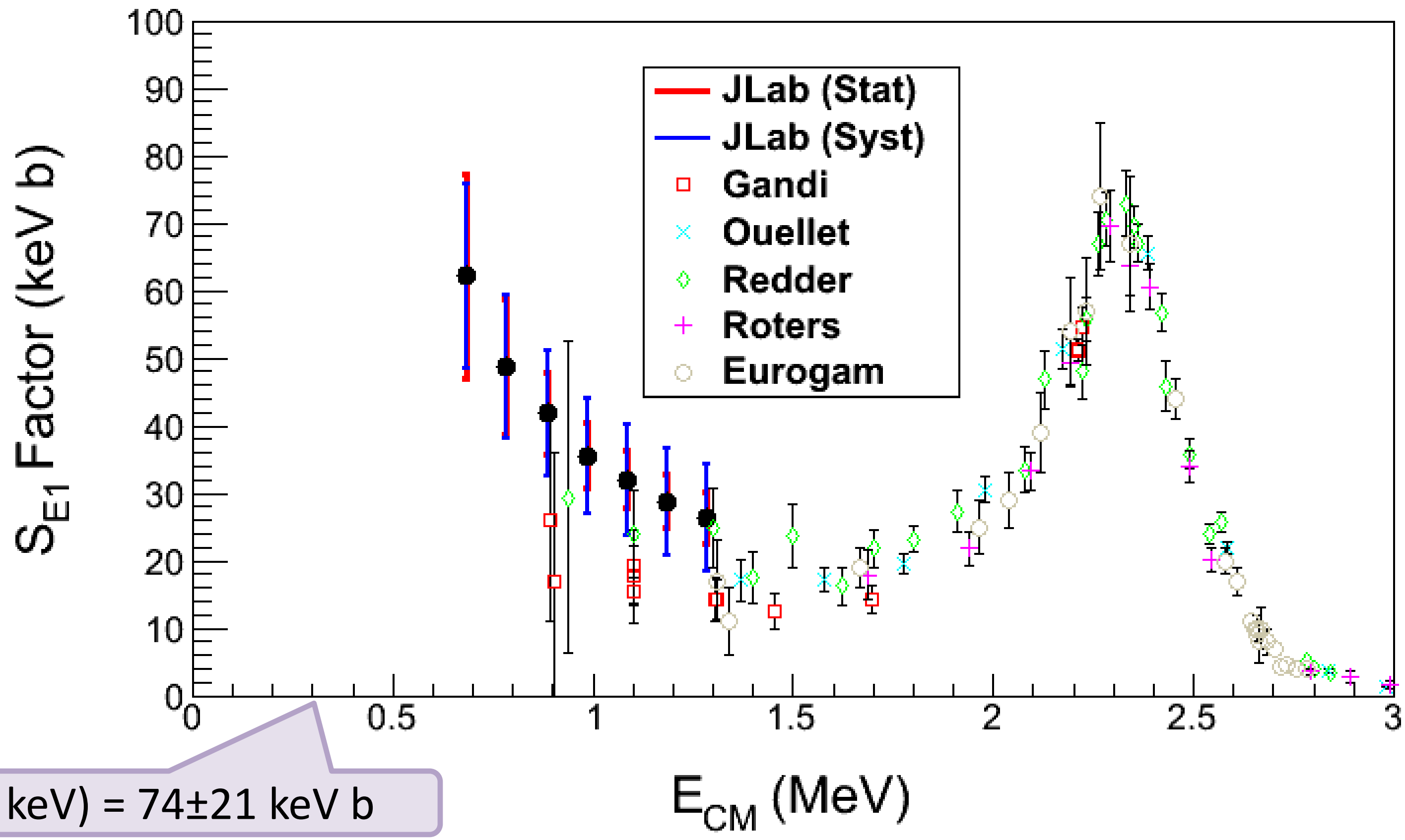
| Electron Beam K. E. | Cross Section (nb) | Sys Error (Energy, %) | Sys Error (Total, %) |
|---|---|---|---|
| 7.9 | 0.046 | 20.1 | 21.9 |
| 8.0 | 0.185 | 19.1 | 22.0 |
| 8.1 | 0.58 | 18.8 | 22.5 |
| 8.2 | 1.53 | 19.1 | 23.8 |
| 8.3 | 3.49 | 19.8 | 25.5 |
| 8.4 | 7.2 | 20.6 | 27.7 |
| 8.5 | 13.6 | 21.7 | 30.2 |

| Electron Beam K. E. | Gamma Energy (MeV) | $E_{CM}$ (MeV) | Cross Section (nb) | $S_{E1}$ Factor (keV b) | Stat Error (%) | Sys Error (%) |
|---|---|---|---|---|---|---|
| 7.9 | 7.85 | 0.69 | 0.046 | 62.2 | 24.5 | 21.9 |
| 8.0 | 7.95 | 0.79 | 0.185 | 48.7 | 20.7 | 22.0 |
| 8.1 | 8.05 | 0.89 | 0.58 | 41.8 | 14.7 | 22.5 |
| 8.2 | 8.15 | 0.99 | 1.53 | 35.5 | 13.8 | 23.8 |
| 8.3 | 8.25 | 1.09 | 3.49 | 32.0 | 13.3 | 25.5 |
| 8.4 | 8.35 | 1.19 | 7.2 | 28.8 | 13.8 | 27.7 |
| 8.5 | 8.45 | 1.29 | 13.6 | 26.3 | 14.8 | 30.2 |

# $^{12}C(\alpha, \gamma)^{16}O$ S-Factor

- Statistical Error: dominated by background subtraction from $^{18}O(\gamma,\alpha)^{14}C$ (depletion = 5,000)

- Systematic Error: dominated by absolute beam energy ($\delta E$ = 0.1%)




$S_{E1}$(300 keV) = 74±21 keV b