# Beamline Design for a Stern-Gerlach Deflection Experiment in CEBAF 1D Spectrometer Line

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## 1 Beamline Optics

This note describes a preliminary beamline design for detection and measurement of the Stern-Gerlach deflection of relativistic 0.5 MeV ($\gamma = 2$) polarized electrons in the CEBAF 1D Spectrometer beamline.

Beamline parameters for the proposed beamline are given in Table 1. The quadrupole layout is shown in Figure 1. By starting design the line elements are symmetric about the center at point C0. Except for slightly different initial conditions, and relaxing the output focusing, the optics plots in the following figures respect this symmetry. To assure the design is practically achievable, the strong quadrupoles are patterned after permanent magnet quadrupoles described in Table III of a paper by Li and Musumeci[1]. The first two

Table 1: Lattice parameters for the Stern-Gerlach beamline in the 1D Spectrometer beamline served by the CEBAF injector. Asterisks indicate quadrupoles whose strengths can be much weaker, and tailored to lead the beam gracefully to the beam dump. The assumed kinetic energy is 500 KeV or, approximately $\gamma = 2$.

| label | longit. pos. | Loc. label | Quad name | Quad length | Bore rad. | Inv.foc. length $q$ | $dB_y/dx$ | $\Delta\theta_{SG}$ | $\sigma_y$ | $B(\sigma)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | cm | | | mm | mm | 1/m | T/m | Å/m | mm | T |
| s0 | 0 | B0 | | | | | | | | |
| s1 | 1.0 | B1 | qBC1 | 8 | 10 | -725.4716 | -263.9 | -1.6234 | 2.84 | -0.750 |
| s2 | 1.2 | B2 | qBC2 | 8 | 10 | 396.0415 | 144.1 | 0.8862 | 1.28 | 0.185 |
| s3 | 1.7 | B3 | qBC3 | 8 | 20 | -181.9351 | -66.17 | -0.4071 | 14.14 | -0.935 |
| s4 | 7.4 | C0 | | | | | | | | |
| s5 | 12.7 | C1 | qCD1* | 8 | 20 | -38.4757 | -14.00 | -0.086 | 14.02 | -0.196 |
| s6 | 13.2 | C2 | qCD2* | 8 | 20 | 7.9208 | 2.88 | 0.017 | 11.32 | 0.033 |
| s7 | 13.4 | C3 | qCD3* | 8 | 20 | 40.0000 | 14.54 | 0.090 | 10.41 | 0.152 |
| s8 | 14.4 | C4 | | | | | | | | |

quadrupoles have substantially smaller bore radii than the rest. This enables their much larger field gradients. Optical properties of the beamline are shown in the following figures.

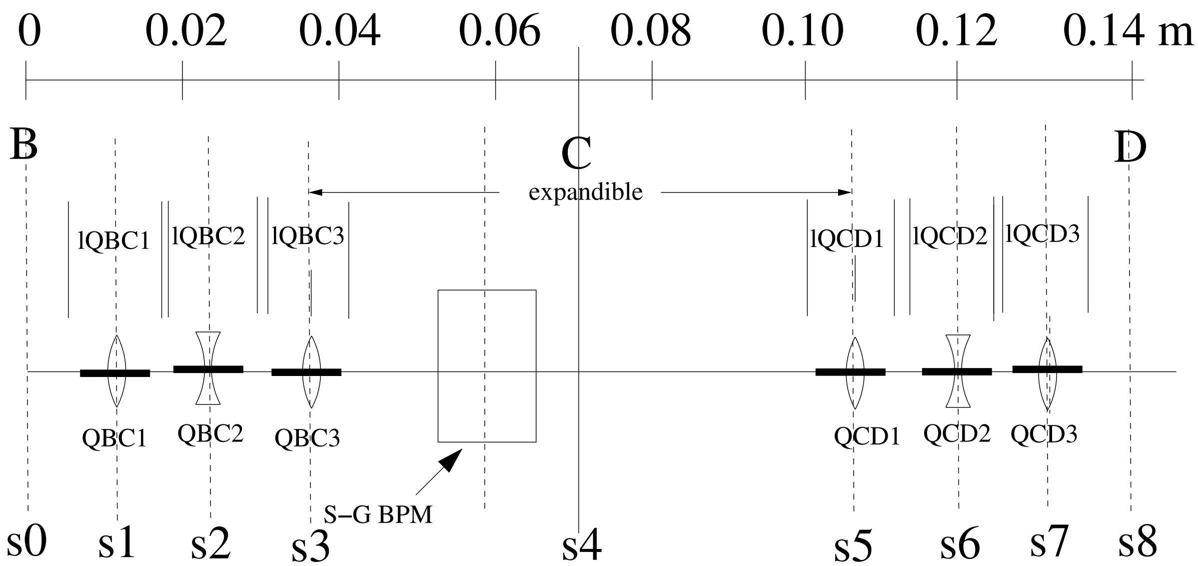


Figure 1: A six-quadrupole beam line for detection and measurement of Stern-Gerlach deflection of a polerized electron beam. The heavy black lines indicating quadrupole lengths are not-to-scale, except, of course, that they cannot overlap. The dimensions on the plot are approximately valid for a 500 KeV (kinetic energy) electron beam. But, by scaling all lengths and quadrupole focal lengths, the same design is also applicable to other energies, for example 5 MeV. In order for the S-G deflection to be purely vertical the quadrupoles have to be "skew", i.e. at 45° relative to "erect".

The basic goal for the beamline optical design is to magnify the Stern-Gerlach displacement without excessively increasing the transverse beam dimensions. Quadrupole doublet design is made ineffective by the fact that the S-G deflections in two nearby, approximately equal, but opposite sign quadrupoles, approximately cancel. The trick to overcoming this cancellation can be understood from Figure 2. By placing the second quadrupole, qBC2, at the beam waist caused by the vertical over-focusing in the first quadrupole, qBC1, the S-G deflection at qBC2 can be cancelled. As a result, the dominant S-G deflection in the line is that caused by qBC1; see Table 2. The design has the further virtue that the beta functions are fairly large at source and at BPM.

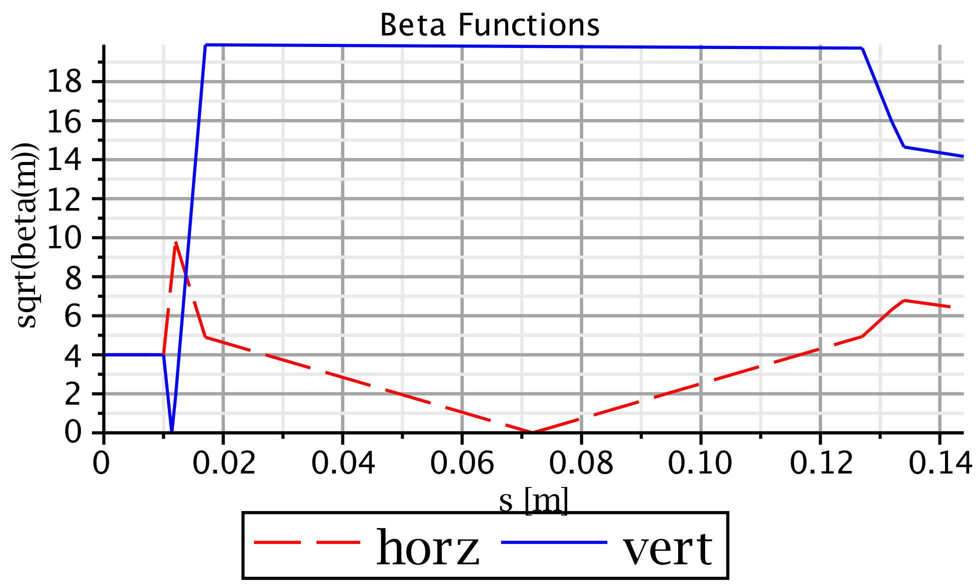


Figure 2: Vertical, $\sqrt{\beta_y(s)}$, and horizontal, $\sqrt{\beta_x(s)}$, beta functions for S-G detection and measurement in the CEBAF 1D Spectrometer Line. In the central region the beta function variations are weak. There is the possibility of some space charge blow-up occurring near the horizontal focus at central point C. But this is expected to be acceptably small, at least partially because of the substantial horizontal dispersion caused by the 30 degree bend at the entrance to the line.

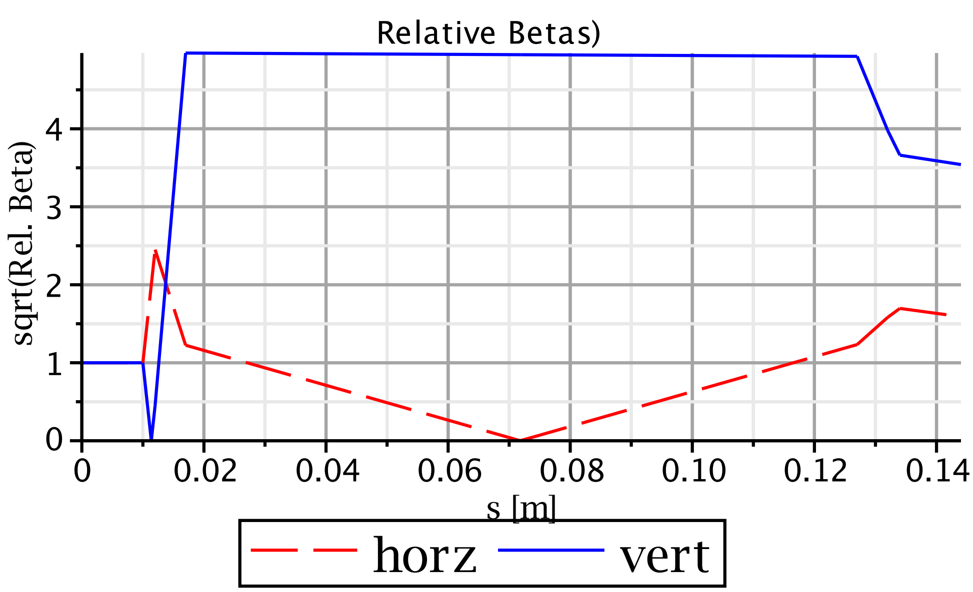


Figure 3: (Square root) beta function ratios $\sqrt{\beta_x(s)/\beta_x(s_0)}$ and $\sqrt{\beta_y(s)/\beta_y(s_0)}$. The initial beta functions, $\beta(s_0)$ are determined by the beam emittances (which vary inversely with the relativistic beam energy factor $\gamma$).

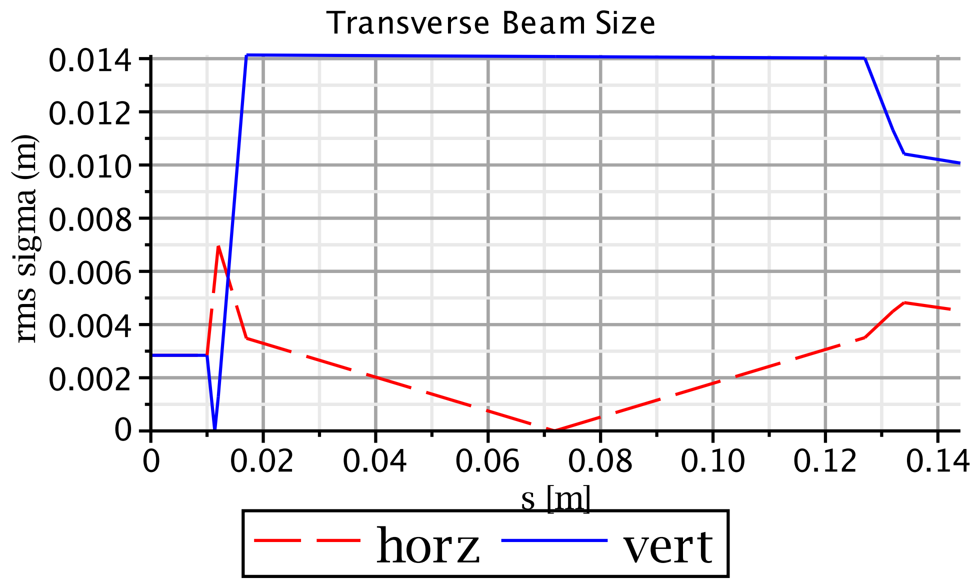


Figure 4: Transverse rms beam sizes as functions of longitudinal position $s$. For designs in this tech-note, the maximum rms beam sizes have been constrained to be less than $\sigma_y = 1.5\,\mathrm{cm}$. If this is impractically large, either the vertical beam emittance will have to be reduced, or the magnitude of the S-G deflection decreased. For first detecting the S-G signal (which has never before been accomplished) the former approach is preferable. But, for eventual precision Stern-Gerlach measurement, the S-G signal can probably be much reduced without seriously impacting the precision. In other words, the optical magnification can be reduced, to reduce the maximum transverse beam dimensions, and optimize the precision.

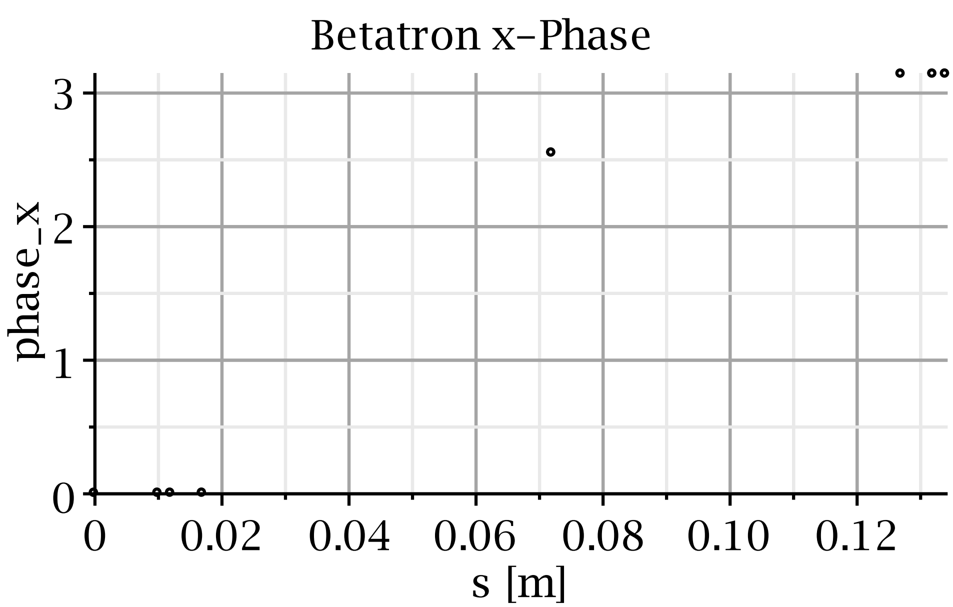


Figure 5: Horizontal betatron phase advance as function of longitudinal position $s$.

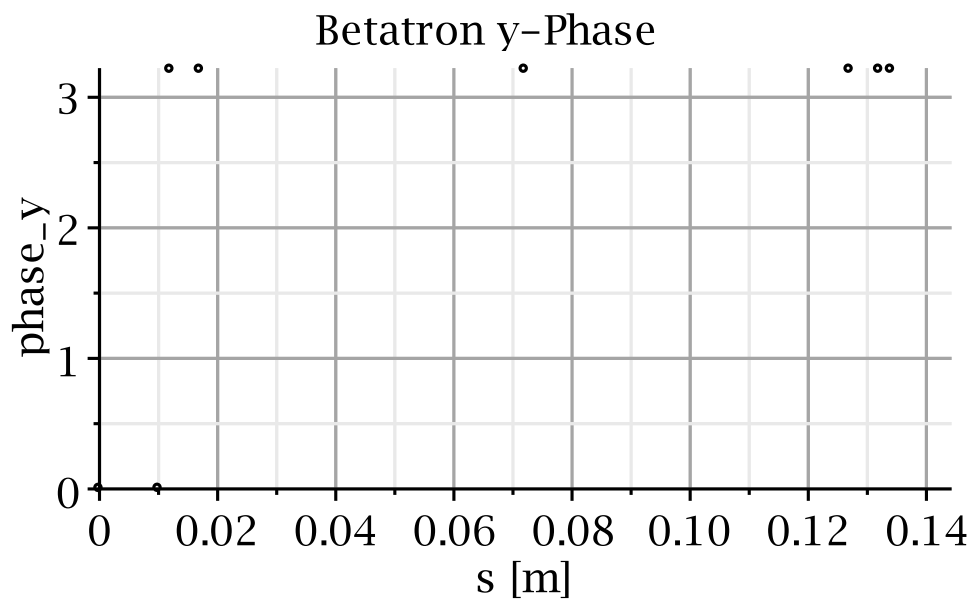


Figure 6: Vertical betatron phase advance $\psi_y(s)$, as function of longitudinal position $s$. Because $\psi_y(s)$ is essentially constant over the central, high-$\beta$ region, the Stern-Gerlach displacement does not vary noticeably over this central region, in spite of the substantial angular deflection $\Delta\theta_y^{SG}$, caused by the (very strong) qBC1 quadrupole. Whatever S-G displacement there is, is mostly already present at the qBC3 quadrupole.

## 2 Calculated Stern-Gerlach Displacement

The ratio of Stern-Gerlach to electromagnetic force is determined by a ratio of coupling constants:

$$\frac{\mu_B/c}{e} = 1.930796 \times 10^{-13}\,\mathrm{m}, \tag{1}$$

where, except for anomalous magnetic moment and sign, Bohr magneton $\mu_B$ is the electron magnetic moment. Because this ratio is not dimensionless, a further inverse length numerical factor, alters the ratio of S-G to electromagnetic deflection angle. This factor can be taken to be a quadrupole strength (i.e. inverse focal length) $q$, which can be quite large numerically (approaching 1000/m, for example); see Table 1.

The Stern-Gerlach deflection in a quadrupole is strictly proportional to the inverse focal length of the quadrupole[2];

$$\boxed{\Delta\theta_y^{SG} = -\frac{\mu_x^*}{ec\beta} q_x, \quad \text{and} \quad \Delta\theta_y^{SG} = \frac{\mu_y^*}{ec\beta} q_y,} \tag{2}$$

These formulas are boxed to emphasize their universal applicability to all cases of polarized beams passing through quadrupoles. For all practical (electron beam) cases $\beta \approx 1$.

The S-G deflection at fixed magnet excitation is proportional to $1/\gamma$. Yet, superficially, these formulas show no *explicit* dependence on $\gamma$. This is only because the angular deflections are expressed in terms of quadrupole inverse focal lengths. For a given quadrupole at fixed quadrupole excitation, the inverse focal length scales as $1/\gamma$. Expressing the S-G deflection in terms of inverse focal lengths has the effect of "hiding" the $1/\gamma$ Stern-Gerlach deflection dependence, which comes from the beam stiffness.

$\mu_x^*$ and $\mu_y^*$ differ from the Bohr magnetron $\mu_B$ only by $\sin\theta$ and $\cos\theta$ factors respectively For a single quadrupole, the Stern-Gerlach-induced angular deflection is

$$\Delta\theta_y^{SG} = (1.93 \times 10^{-13}\,\mathrm{m})\, q_y. \tag{3}$$

The transverse displacement $\Delta y_j$ at downstream location "j" caused by angular displacement $\Delta\theta_{y,i}$ at upstream location "i" is given by

$$\Delta_{y,j} = q_y\,(1.93 \times 10^{-13}\,\mathrm{m})\,\sqrt{\beta_{y,j}\beta_{y,i}}\,\sin(\psi_{y,j} - \psi_{y,i}). \tag{4}$$

where $\psi_{y,j} - \psi_{y,i}$ is the vertical betatron phase advance from "i" to "j". The lattice optics initial conditions will not be very well known initially. Yet the optimal beamline design is quite sensitive to the beam conditions. The calculations in this note have assumed the following plausible initial beam conditions (for both horizontal and vertical coordinates):

$$\begin{aligned} \epsilon &= \frac{1.0 \times 10^{-6}\,\mathrm{m}}{\gamma}, \\ \sigma^{B0} &= \frac{0.004\,\mathrm{m}}{\sqrt{\gamma}}, \\ \beta^{B0} &= \frac{\sigma^2}{\epsilon}, \quad \alpha^{B0} = 0, \quad \psi^{B0} = 0. \end{aligned} \tag{5}$$

```
epsilon_x := 1.0e-6/gamrel;
epsilon_y := 1.0e-6/gamrel;
sigmafac := 1/sqrt(gamrel);
sigmaB0x := 0.004*sigmafac;  # sigmaB0x := 0.004*sigmafac;
sigmaB0y := 0.004*sigmafac;  # sigmaB0y := 0.004*sigmafac;
betB0x := sigmaB0x^2/epsilon_x;
betB0y := sigmaB0y^2/epsilon_y;
alfB0x := 0.0:  alfB0y := 0.0:
phB0x := 0.0:   phB0y := 0.0:
betC0y := 0.001;
```

The resulting S-G deflections are shown in Table 2. As explained earlier the S-G displacement

Table 2: .Stern-Gerlach displacements, measured in Å units, at points along the beamline, for kinetic energy $K_e = 500\,\mathrm{KeV}$.

| S-G deflec. source | displacement at C0 |
|---|---|
| | Å |
| qBC1 | 8.832 |
| qBC2 | 0.009 |
| qBC3 | -0.022 |
| qCD1 | 0 |
| qCD2 | 0 |
| total | 8.82 |

is essentially constant over the central region. Stretching the central region, even by a large amount, has little effect on the S-G displacement. This can allow allow the S-G detection BPM to be long, to increase their sensitivity, or even multiple, to lower the noise floor.

# 3 Uncertainty and Conclusions

The greatest uncertainty in the calculation concerns Eqs. (5), and the corresponding lines of code listed below these equations. To magnify the Stern-Gerlach deflection one wants the vertically-deflecting quadrupole to be strong. This automatically causes $\beta_y$ to increase, which increases the beam height. Accepting the limitation that the rms beam height cannot exceed 15 mm, this limits the downstream S-G displacement. In detail this limitation depends on the initial beta function/emittance/beam height assumptions. The entries in Eqs. (5) are subject to change as the beamline is tuned up. Comments on optimizing the beamline have been given in the caption to Figure 4.

The Stern-Gerlach energy dependence has been much discussed in the past. The importance of the transverse beam size has not previously, as far as I know, been properly appreciated in those discussions. It is now my opinion that, as long as the transverse beam dimensions are dominated by adiabatic damping (with increased energy), the quadrupole length/strength scaling can be maintained, and the transverse aperture limit is independent of energy, that the achievable S-G-induced betatron beam deflection is more or less independent of energy.

As far as the proof-of-principle test at CEBAF, the most convenient energy appears to be at 500 KeV, but this is for reasons of economy and accessibility, not because the S-G signal is strongest at low energy.

For the assumed electron beam parameters, I have been unable to produce S-G betatron amplitude greater than 1 nanometer. As I have argued previously and repeatedly, especially with Reza's suggested toggling-polarization beam preparation, with further low frequency beam polarization modulation, and with accurate BPM centering, it should not be difficult to isolate this Stern-Gerlach signal from the many spurious sources of BPM excitation.

# References


[1] R. Li and P. Musumeci, *Single-Shot MeV Transmission Electron Microscopy with Picosecond Temporal Resolution,* Physical Review Applied 2, 024003, 2014.

[2] R. Talman, *Relativistic Stern-Gerlach Deflection,* arXiv 1611.0380 [physics.acc-ph], 2016