Beamline Design for a Stern-Gerlach Deflection Experiment in CEBAF 1D Spectrometer Line

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1 Beamline Optics

This note describes a preliminary beamline design for detection and measurement of the Stern-Gerlach deflection of relativistic 0.5 MeV ($\gamma = 2$) polarized electrons in the CEBAF 1D Spectrometer beamline.

Beamline parameters for a first pass design of the proposed beamline are given in Table 1. The quadrupole layout is shown in Figure 1. Essentially the same quadrupole layout will be used in a later section to give a much improved design. By starting design the line elements are symmetric about the center at point C0. Except for slightly different initial conditions, and relaxing the output focusing, the optics plots in the following figures respect this symmetry. To assure the design is practically achievable, the strong quadrupoles are

Table 1: Lattice parameters for the Stern-Gerlach beamline in the 1D Spectrometer beamline served by the CEBAF injector. Asterisks indicate quadrupoles whose strengths can be much weaker, and tailored to lead the beam gracefully to the beam dump. The assumed kinetic energy is 500 KeV or, approximately $\gamma = 2$.

label	longit. pos.	Loc. label	Quad name	Quad length	Bore rad.	Inv.foc. length q	dB_y/dx	$\Delta\theta_{SG}$	σ_y	$B(\sigma)$
	cm			mm	mm	1/m	T/m	Å/m	mm	T
s0	0	B0								
s1	1.0	B1	qBC1	4	10	-362.736	-263.9	-0.8117	2.84	-0.750
s2	1.4	B2	qBC2	4	10	195.648	142.3	0.4378	1.28	0.185
s3	2.4	B3	qBC3	4	20	-90.9039	-66.1	-0.2034	14.14	-0.935
s4	13.4	C0								
s5	24.4	C1	qCD1*	4	20	-19.238	-7.00	*	*	
s6	25.4	C2	qCD2*	4	20	3.96	1.44	*	*	
s7	25.8	C3	qCD3*	4	20	20.00	7.27	*	*	
s8	26.8	C4								

patterned after permanent magnet quadrupoles described in Table III of a paper by Li and Musumeci[1]. The first two quadrupoles have substantially smaller bore radii than the rest. This enables their much larger field gradients. Optical properties of the beamline are shown in the following figures.

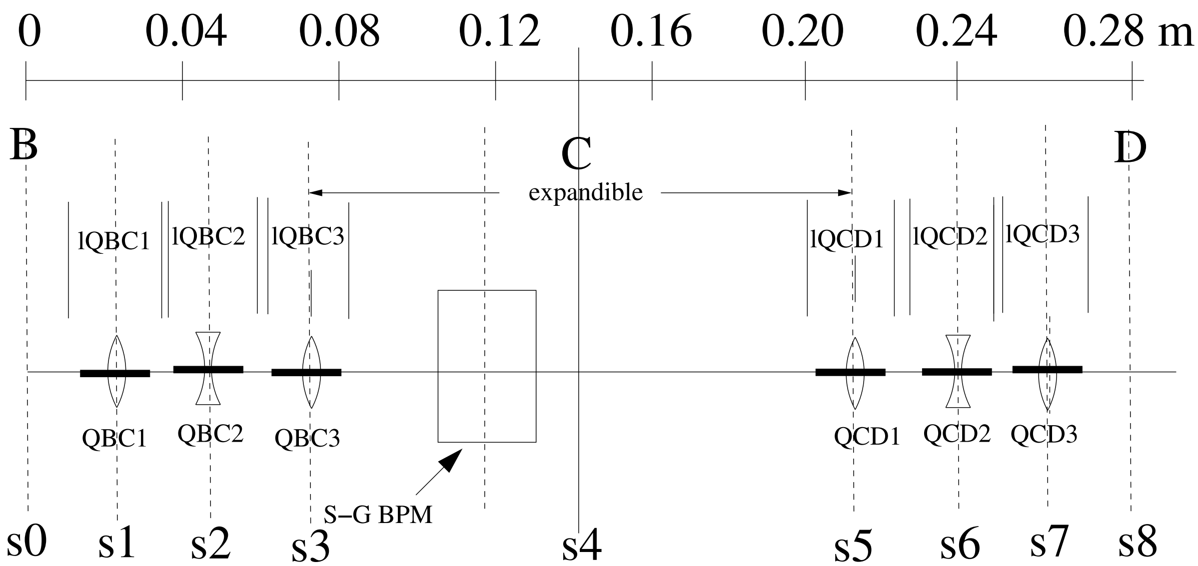

Figure 1: A six-quadrupole beam line for detection and measurement of Stern-Gerlach deflection of a polerized electron beam. The heavy black lines indicating quadrupole lengths are not-to-scale, except, of course, that they cannot overlap. The dimensions on the plot are approximately valid for a 500 KeV (kinetic energy) electron beam. But, by scaling all lengths and quadrupole focal lengths, the same design is also applicable to other energies, for example 5 MeV. In order for the S-G deflection to be purely vertical the quadrupoles have to be "skew", i.e. at 45° relative to "erect".

The basic goal for the beamline optical design is to magnify the Stern-Gerlach displacement without excessively increasing the transverse beam dimensions. Quadrupole doublet design is made ineffective by the fact that the S-G deflections in two nearby, approximately equal, but opposite sign quadrupoles, approximately cancel. The trick to overcoming this cancellation can be understood from Figure 2. By placing the second quadrupole, qBC2, at the beam waist caused by the vertical over-focusing in the first quadrupole, qBC1, the S-G deflection at qBC2 can be cancelled. As a result, the dominant S-G deflection in the line is that caused by qBC1; see Table 2. The design has the further virtue that the beta functions are fairly large at source and at BPM.

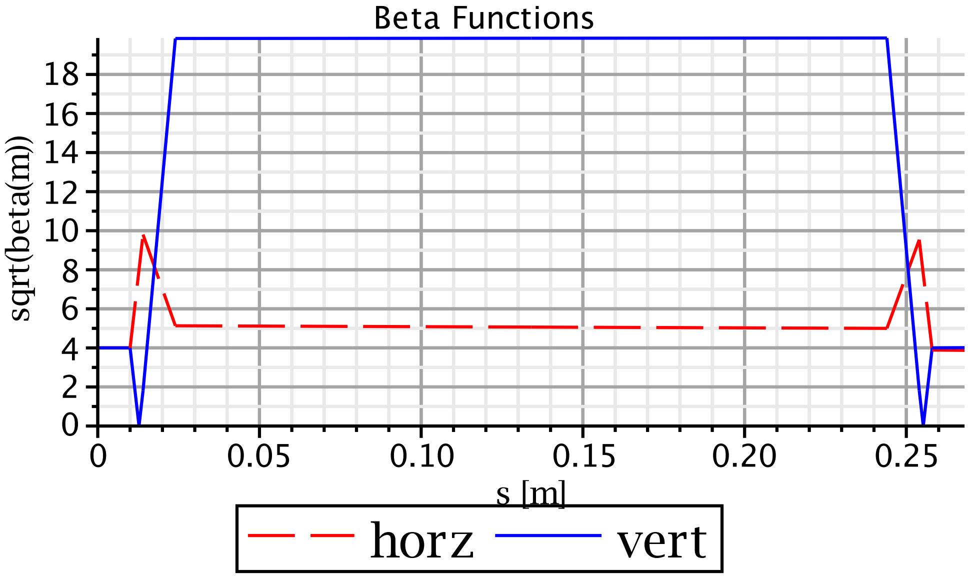

Figure 2: Vertical, $\sqrt{\beta_y(s)}$, and horizontal, $\sqrt{\beta_x(s)}$, beta functions for S-G detection and measurement in the CEBAF 1D Spectrometer Line. In the central region the beta function variations are weak. There is the possibility of some space charge blow-up occurring near the horizontal focus at central point C. But this is expected to be acceptably small, at least partially because of the substantial horizontal dispersion caused by the 30 degree bend at the entrance to the line.

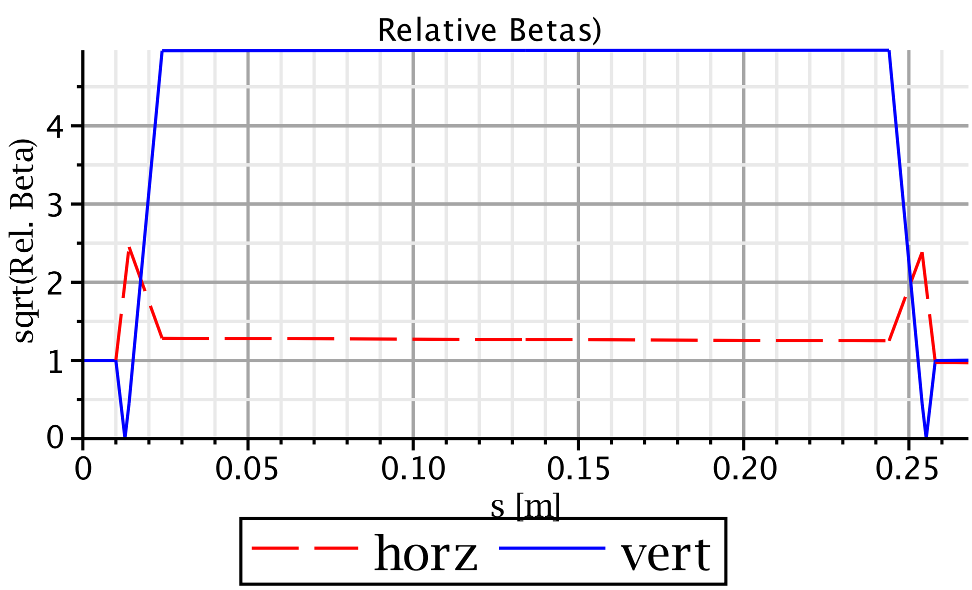

Figure 3: (Square root) beta function ratios $\sqrt{\beta_x(s)/\beta_x(s_0)}$ and $\sqrt{\beta_y(s)/\beta_y(s_0)}$. The initial beta functions, $\beta(s_0)$ are determined by the beam emittances (which vary inversely with the relativistic beam energy factor γ).

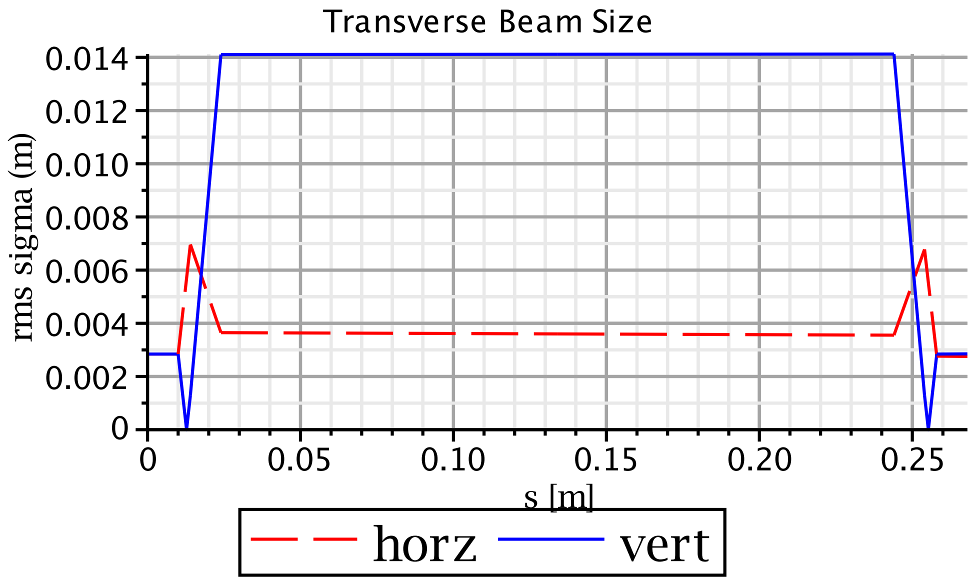

Figure 4: Transverse rms beam sizes as functions of longitudinal position s. For designs in this tech-note, the maximum rms beam sizes have been constrained to be less than $\sigma_y = 1.5\,\mathrm{cm}$. If this is impractically large, either the vertical beam emittance will have to be reduced, or the magnitude of the S-G deflection decreased. For first detecting the S-G signal (which has never before been accomplished) the former approach is preferable. But, for eventual precision Stern-Gerlach measurement, the S-G signal can probably be much reduced without seriously impacting the precision. In other words, the optical magnification can be reduced, to reduce the maximum transverse beam dimensions, and optimize the precision.

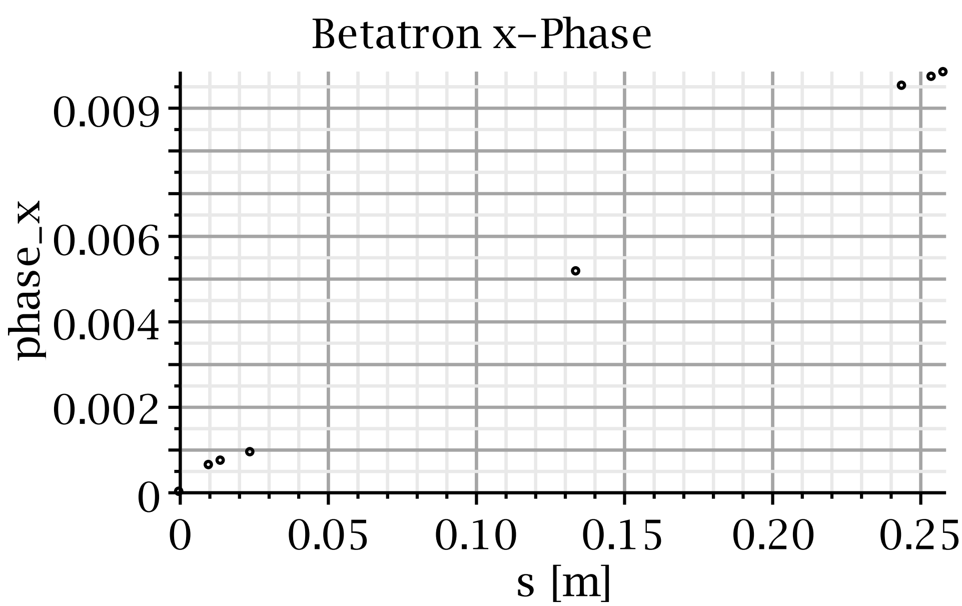

Figure 5: Horizontal betatron phase advance as function of longitudinal position s.

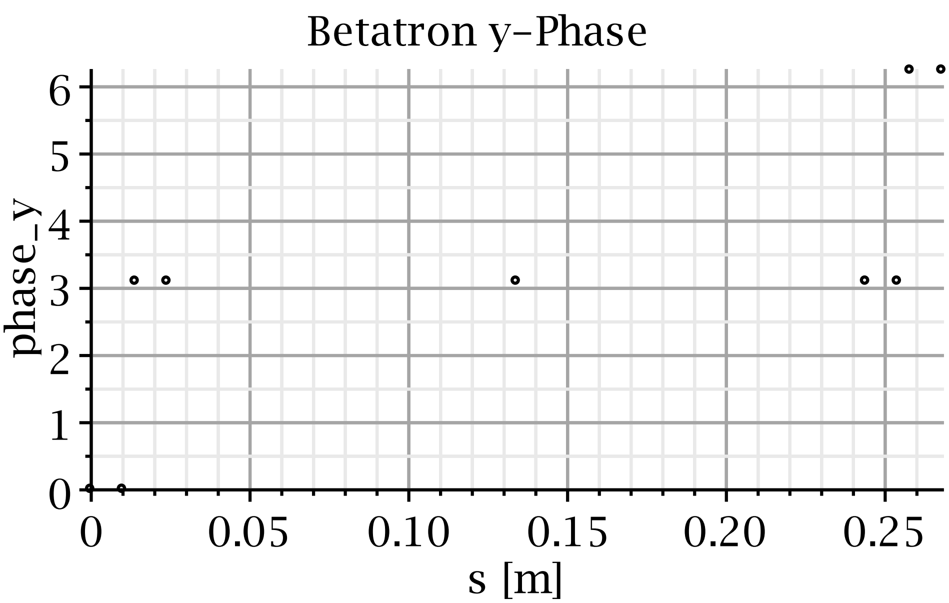

Figure 6: Vertical betatron phase advance $\psi_y(s)$, as function of longitudinal position s. Because $\psi_y(s)$ is essentially constant over the central, high-β region, the Stern-Gerlach displacement does not vary noticeably over this central region, in spite of the substantial angular deflection $\Delta\theta_y^{SG}$, caused by the (very strong) qBC1 quadrupole. Whatever S-G displacement there is, is mostly already present at the qBC3 quadrupole.

2 Calculated Stern-Gerlach Displacement

The ratio of Stern-Gerlach to electromagnetic force is determined by a ratio of coupling constants:

$$\frac{\mu_B/c}{e} = 1.930796 \times 10^{-13}\,\mathrm{m}, \tag{1}$$

where, except for anomalous magnetic moment and sign, Bohr magneton μ_B is the electron magnetic moment. Because this ratio is not dimensionless, a further inverse length numerical factor, alters the ratio of S-G to electromagnetic deflection angle. This factor can be taken to be a quadrupole strength (i.e. inverse focal length) q, which can be quite large numerically (approaching 1000/m, for example); see Table 1.

The Stern-Gerlach deflection in a quadrupole is strictly proportional to the inverse focal length of the quadrupole[2];

$$\boxed{\Delta\theta_y^{SG} = -\frac{\mu_x^*}{ec\beta}q_x, \quad \text{and} \quad \Delta\theta_y^{SG} = \frac{\mu_y^*}{ec\beta}q_y,} \tag{2}$$

These formulas are boxed to emphasize their universal applicability to all cases of polarized beams passing through quadrupoles. For all practical (electron beam) cases $\beta \approx 1$.

The S-G deflection at fixed magnet excitation is proportional to $1/\gamma$. Yet, superficially, these formulas show no *explicit* dependence on γ. This is only because the angular deflections are expressed in terms of quadrupole inverse focal lengths. For a given quadrupole at fixed quadrupole excitation, the inverse focal length scales as $1/\gamma$. Expressing the S-G deflection in terms of inverse focal lengths has the effect of "hiding" the $1/\gamma$ Stern-Gerlach deflection dependence, which comes from the beam stiffness.

μ_x^* and μ_y^* differ from the Bohr magnetron μ_B only by $\sin\theta$ and $\cos\theta$ factors respectively For a single quadrupole, the Stern-Gerlach-induced angular deflection is

$$\Delta\theta_y^{SG} = (1.93 \times 10^{-13}\,\mathrm{m})\, q_y. \tag{3}$$

The transverse displacement Δy_j at downstream location "j" caused by angular displacement $\Delta\theta_{y,i}$ at upstream location "i" is given by

$$\Delta_{y,j} = q_y\,(1.93 \times 10^{-13}\,\mathrm{m})\,\sqrt{\beta_{y,j}\beta_{y,i}}\,\sin(\psi_{y,j} - \psi_{y,i}). \tag{4}$$

where $\psi_{y,j} - \psi_{y,i}$ is the vertical betatron phase advance from "i" to "j". The lattice optics initial conditions will not be very well known initially. Yet the optimal beamline design is quite sensitive to the beam conditions. The calculations in this note have assumed the following plausible initial beam conditions (for both horizontal and vertical coordinates):

$$\begin{aligned} \epsilon &= \frac{1.0 \times 10^{-6}\,\mathrm{m}}{\gamma}, \\ \sigma^{B0} &= \frac{0.004\,\mathrm{m}}{\sqrt{\gamma}}, \\ \beta^{B0} &= \frac{\sigma^2}{\epsilon}, \quad \alpha^{B0} = 0, \quad \psi^{B0} = 0. \end{aligned} \tag{5}$$

```
epsilon_x := 1.0e-6/gamrel;
epsilon_y := 1.0e-6/gamrel;
sigmafac := 1/sqrt(gamrel);
sigmaB0x := 0.004*sigmafac;  # sigmaB0x := 0.004*sigmafac;
sigmaB0y := 0.004*sigmafac;  # sigmaB0y := 0.004*sigmafac;
betB0x := sigmaB0x^2/epsilon_x;
betB0y := sigmaB0y^2/epsilon_y;
alfB0x := 0.0:  alfB0y := 0.0:
phB0x := 0.0:   phB0y := 0.0:
```

The resulting S-G deflections are shown in Table 2. As explained earlier the S-G displacement

Table 2: .Stern-Gerlach displacements, measured in Å units, at points along the beamline, for kinetic energy $K_e = 500\,\mathrm{KeV}$.

S-G deflec. source	displacement at C0
	Å
qBC1	2.767
qBC2	0.009
qBC3	-0.022
qCD1	0
qCD2	0
total	2.754

is essentially constant over the central region. Stretching the central region, even by a large amount, has little effect on the S-G displacement. This can allow allow the S-G detection BPM to be long, to increase their sensitivity, or even multiple, to lower the noise floor.

3 Beam Demagnification with Spin Reversing Solenoid

The line described so far has the serious disadvantage that the transverse beam dimensions are very large at the location of the Stern-Gerlach sensing beam position monitor. By constraining the beamline optics to be symmetric about central point C0 the beam size can be restored to its original slender dimensions.

Unfortunately, having the quadrupole configuration mirror symmetric will exactly cancel the S-G induced beam displacement at the beamline exit. A solenoid placed in the central drift region, and adjusted to a strength that reverses the spin orientations overcomes this cancellation problem. The transverse spin precession in a solenoid can be expressed in term of the electron cyclotron frequency, which is given by

$$\frac{e}{m_e} = 1.759\,820\,150 \times 10^{11}\ \mathrm{s}^{-1}\mathrm{T}^{-1}. \tag{6}$$

In a solenoid of length L_s, with magnetic field B_π adjusted to give a π-rotation for an electron with speed v, one has

$$\pi = \Omega_\pi \frac{L_s}{v} = \frac{1}{\gamma} \frac{e}{m_e} B_\pi \frac{L_s}{v}, \tag{7}$$

or

$$B_\pi = \gamma\pi \frac{1}{e/m_e} \frac{L_s}{v} \overset{\text{e.g.}}{=} 2\pi \frac{1}{1.760 \times 10^{11}} \frac{0.08}{0.863 \times 3 \times 10^8} = 0.1142\,\mathrm{T}. \tag{8}$$

As a result of the spin flip, instead of adding destructively, the S-G displacements induced by the input and output quadrupoles add constructively at the exit BPM. The beam optics of the revised beamline are shown in Figure 8. The slightly revised are parameters

```
s0 := 0;
s1 := 0.01;
s2 := 0.0120;
s3 := 0.0170;
s4 := 0.0720;
s5 := 0.1270;
s6 := 0.1320;
s7 := 0.1340;
s8 := 0.1440;

qBC1 := -725.4716256;
qBC2 :=  396.0415150*0.99*0.998;
qBC3 := -183.2176730*0.993*0.9993;

qCD1 := qBC3;
qCD2 := qBC2;
qCD3 := qBC1;
```

Though the net effect of the spin-flip is to double the previously calculated S-G betatron displacement, the exit demagnification reduces the S-G displacement by the same factor by which the beam dimensions are reduced. In this case, from Figure 3, the demagnification factor is 5.0. Copying from Table 2, the S-G betatron amplitude at the exit BPM will be $2.754 \times 2/5 = 1.10\,\text{Å}$.

The focusing effect of the solenoid has been neglected in the calculation.

4 Uncertainty and Conclusions

As just implied, the optical design is in a premature state, for example because the solenoid focusing has been neglected. Probably more important is that the quadrupoles have been treated as having zero length, as regards their focusing effect (though not as regards their maximum field gradients).

Two schemes have been presented. The second design, because of its small transverse beam size, is probably required if standard antenna BPM's are to be used. For a resonant cavity BPM it seems likely that large transverse beam size is not a serious impediment. In that case the increased signal level in the first design may be favored. Actually it would

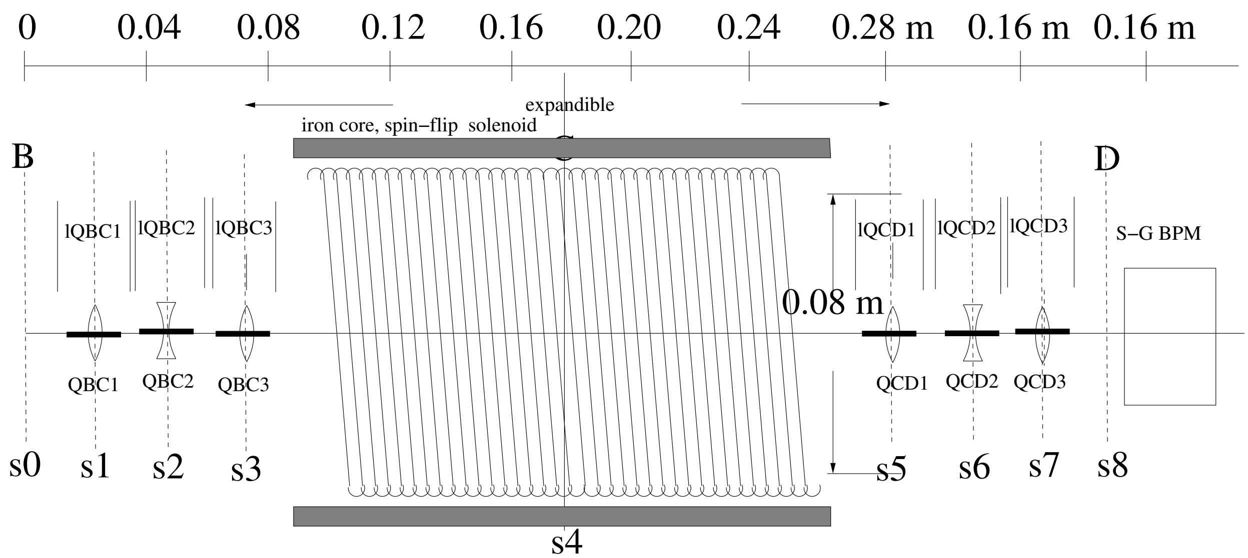

Figure 7: With the central drift region expanded in length, an iron-core solenoid magnet is now included in the previously-described beamline. The solenoid has the important role of reversing the polarity of transversely polarized electrons. The quadrupole strengths are constrained to be exactly symmetric about center position C0, and to return the transverse beam sizes to their values at the beamline entrance. With the beam again slender, its transverse position can be measured with conventional BPM's.

be practical to apply both methods. By shortening the solenoid, or further stretching the central region, one could use both resonant BPM in the central region and antenna BPM in the output region.

In partial compensation for the factor of 2/5 loss of measureable S-G amplitude with the second design, it can be noted that the presence of the spin-flipping solenoid provides a further powerful discrimination against spurious BPM response to beam charge caused by equipment imperfections. With solenoid powered off, the S-G signal detected at harmonics of the beam polarization frequency (i.e. at 750 MHz) should vanish.

The greatest uncertainty in the calculation concerns Eqs. (5), and the corresponding lines of code listed below these equations. To magnify the Stern-Gerlach deflection one wants the vertically-deflecting quadrupole to be strong. This automatically causes β_y to increase, which increases the beam height. Accepting the limitation that the rms beam height cannot exceed 15 mm, this limits the downstream S-G displacement. In detail this limitation depends on the initial beta function/emittance/beam height assumptions. The entries in Eqs. (5) are subject to change as the beamline is tuned up. Comments on optimizing the beamline have been given in the caption to Figure 4.

The Stern-Gerlach energy dependence has been much discussed in the past. The importance of the transverse beam size has not previously, as far as I know, been properly appreciated in those discussions. It is now my opinion that, as long as the transverse beam dimensions are dominated by adiabatic damping (with increased energy), the quadrupole length/strength scaling can be maintained, and the transverse aperture limit is independent of energy, that the achievable S-G-induced betatron beam deflection is more or less

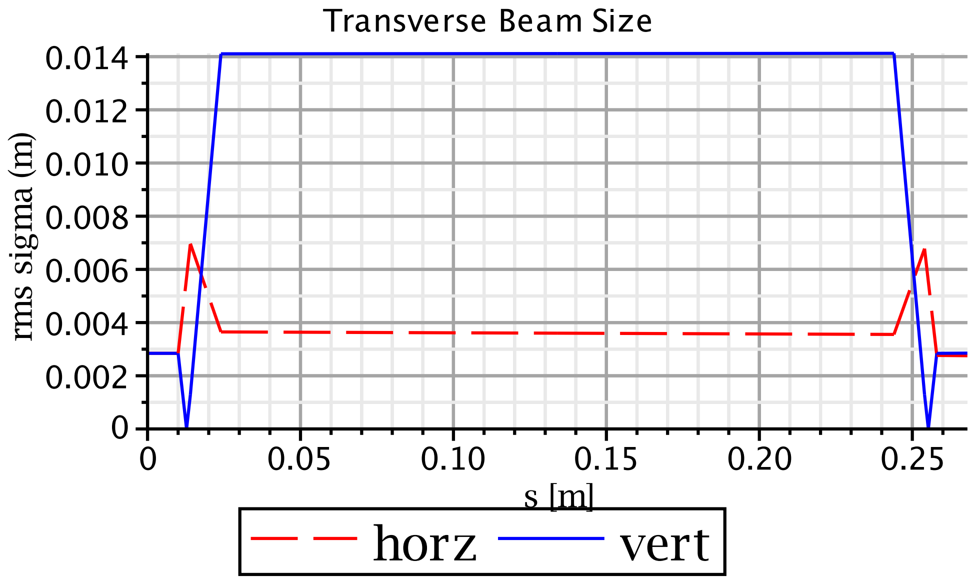

Figure 8: Beamline redesigned to allow for the insertion of a spin-flip solenoid. Its purpose is to preserve the Stern-Gerlach signal that would otherwise be cancelled by the exit focusing.

independent of energy.

As far as the proof-of-principle test at CEBAF, the most convenient energy appears to be at 500 KeV, but this is for reasons of economy and accessibility, not because the S-G signal is undetectable at higher energies.

For the assumed electron beam parameters, I have been unable to produce S-G betatron amplitude greater than 1 nanometer. As I have argued previously and repeatedly, especially with Reza's suggested toggling-polarization beam preparation, with further low frequency beam polarization modulation, and with accurate BPM centering, it should not be difficult to isolate this Stern-Gerlach signal from the many spurious sources of BPM excitation.

References

[1] R. Li and P. Musumeci, *Single-Shot MeV Transmission Electron Microscopy with Picosecond Temporal Resolution,* Physical Review Applied 2, 024003, 2014.

[2] R. Talman, *Relativistic Stern-Gerlach Deflection,* arXiv 1611.0380 [physics.acc-ph], 2016