Understanding the focusing of charged particle beams in a solenoid magnetic field

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The focusing of a charged particle beam in a solenoid is typically explained by invoking the concept of a Larmour frame and using Busch's theorem. Often, there is some confusion about how a uniform magnetic field of a long solenoid focuses the electron beam because it is generally understood that a uniform magnetic field can only guide charged particles. We perform a simple analysis of the dynamics of a charged particle beam in a solenoid and emphasize an intuitive understanding of some of the interesting features. © *2009 American Association of Physics Teachers.*
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I. INTRODUCTION

A solenoid is often used to focus charged particle beams in the low energy section of accelerators[1] and other devices such as electron microscopes.[2] The focusing of charged particle beams in a solenoid is explained in textbooks[3,4] using Busch's theorem,[5] which is a statement of conservation of canonical angular momentum in the axisymmetric magnetic field of the solenoid. In a solenoid, the longitudinal magnetic field on the axis is peaked at the center of the solenoid, decreases toward the ends, and approaches zero far away from the solenoid. In contrast, the radial magnetic field is peaked near the ends of the solenoid. In a simple model, the longitudinal magnetic field can be assumed to be zero outside the solenoid and uniform inside it. When a charged particle enters from the field-free region to the region of uniform magnetic field in a solenoid, it starts rotating with the Larmour frequency, which equals half the cyclotron frequency in the uniform magnetic field. The dynamics of the charged particle is easily analyzed in the Larmour frame, which rotates with the Larmour frequency around the axis of the solenoid. In the Larmour frame, the centrifugal force is half of the Lorentz force, and there is a net focusing force toward the axis of the solenoid.

There is often some confusion about why a charged particle does not rotate with the cyclotron frequency inside the solenoid, as is the case in a uniform magnetic field. It is also confusing how the uniform field in a solenoid provides focusing because a charged particle in a uniform magnetic field has a helical trajectory with a constant radius known as the Larmour radius. The answer to these questions often is buried by the formal analysis in textbooks.

In this paper, we take a fresh look at the problem and perform a simple analysis. We use simple geometric concepts to understand some of the interesting aspects of the dynamics of charged particle beams in a solenoid magnetic field, including the focusing of charged particle beams. The emphasis in this paper is on developing a physical understanding of different aspects of dynamics using simple mathematics.

In Sec. II we analyze the beam dynamics of a charged particle beam in an axisymmetric magnetic field. In Sec. III we discuss some special cases. We first apply this analysis to a thin lens and derive the formula for its focal length. We then apply this analysis to the case where an electron beam generated in a region of uniform magnetic field subsequently exits this region to enter a field-free region. We also discuss the effect of Coulomb repulsion on the beam dynamics. We present some conclusions in Sec. IV.

II. BEAM DYNAMICS IN SOLENOID

Before we discuss the dynamics of a charged particle beam in a solenoid magnetic field, we state the assumptions. First, we ignore the space-charge force, that is, we neglect the Coulomb repulsion between charged particles. Second, we assume a cold beam, that is, the particles' initial transverse velocity is zero. In general, both assumptions are violated in reality. Also, particles will have nonzero transverse velocities, which will have a random distribution inside the beam. We will discuss the effect of removing these assumptions later in this paper.

For simplicity, we assume that the beam is cylindrical with a uniform distribution of particles. All the particles are assumed to have an initial velocity v_z along the z-axis. The components of the axisymmetric magnetic field of a solenoid are given by[4]

$$B_z(r,z) = B(z) - \frac{r^2}{4}B''(z) + \cdots, \tag{1}$$

$$B_r(r,z) = -\frac{r}{2}B'(z) + \frac{r^3}{16}B'''(z) + \cdots, \tag{2}$$

where z is the distance along the solenoid axis, r is the radial distance from the solenoid axis, and the prime denotes a derivative with respect to z. The coordinate system used is shown in Fig. 1. Under the paraxial approximation, we will keep only terms up to the first order term in r. We assume that $B(z)=B_0$ for $0 < z < L$ and $B(z)=0$ otherwise. We are thus assuming that $B(z)$ abruptly drops to zero at the ends. We will show later that our analysis can be generalized to an arbitrary variation in $B(z)$. From Eq. (2) the expression for the magnetic field in this case in the paraxial approximation can be written as
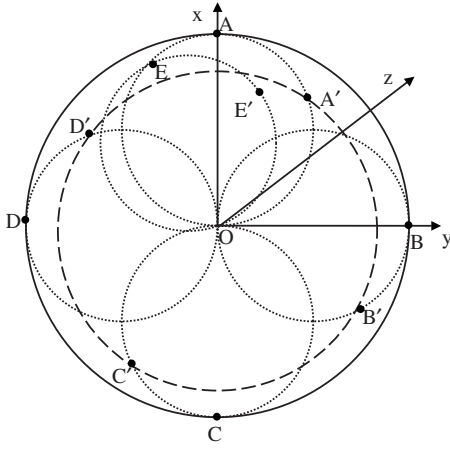
$$B_z = B_0[u(z) - u(z - L)], \tag{3}$$

Fig. 1. Pictorial explanation of the focusing of a charged particle beam in a solenoid. The solid curve shows the periphery of the electron beam when it enters the solenoid. The dashed curve shows the periphery of the electron beam after it travels some distance in the solenoid. The dotted curves show the trajectories of individual electrons. The coordinate axis is also shown.

$$B_r = -\frac{r}{2}B_0[\delta(z) - \delta(z-L)], \tag{4}$$

where $u(z)=1$ for $z>0$ and $u(z)=0$, otherwise. Here, $\delta(z)$ is the Dirac delta function.

We define three regions corresponding to $z<0$ (region I), $0<z<L$ (region II), and $z>L$ (region III). The trajectory of the charged particle will be a straight line in regions I and III because these regions are field-free. The trajectory will be helical in region II because it is a region of uniform magnetic field. We have to match the trajectory at the boundary between regions I and II and between regions II and III. At the boundary between regions I and II, the radial magnetic field is a Delta function as seen from Eq. (4), which gives an impulse in the azimuthal direction. The force in the azimuthal direction is given by $-ev_z B_r(z)$, where e is the magnitude of the electronic charge. Due to this impulse, the increment Δv_θ in the azimuthal velocity of the electron as it crosses this boundary is given by

$$\Delta v_\theta = r_0 \frac{eB_0}{2\gamma m}, \tag{5}$$

where r_0 is the radial coordinate of the particle when it enters region II, m is the electron's rest mass, and $\gamma = 1/\sqrt{1-v_z^2/c^2}$ is the usual Lorentz factor. The cyclotron frequency is defined as $\omega_c = eB_0/\gamma m$ and the Larmour frequency is defined as $\omega_L = eB_0/2\gamma m$. We can therefore write $\Delta v_\theta = r_0 \omega_L$.

The radial component of the velocity will remain continuous when the particle enters region II from region I. We assume that $v_r=0$ in region I. Hence, $v_r=0$ at $z=0$ in region II. Note that when the particle enters region II, the longitudinal component of the velocity will also change because the transverse component changes and the total kinetic energy is conserved. However, in the paraxial approximation $v_\perp/v_z \ll 1$, where $v_\perp$ is the transverse component of the particle velocity. Hence, the change in v_z will be negligible here, and we ignore this change. With the transverse velocity given by Eq. (5), the particle will have a helical trajectory inside region II with the radius R_c given by
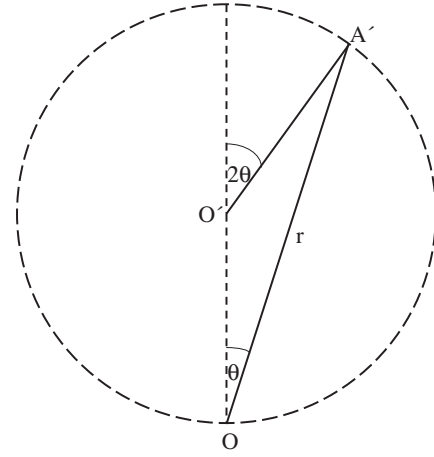

Fig. 2. Illustration of the relation between the Larmour frequency and the cyclotron frequency. Point O is on the axis of the solenoid and point O′ is the center of the trajectory.

$$R_c = \left| \frac{\gamma m v_\theta}{eB_0} \right| = \frac{r_0}{2}. \tag{6}$$

Equation (6) tells us that for every off-axis particle that enters the magnetic field, the radius of curvature inside region II is half of the initial radial displacement of the particle from the solenoid axis. The projections of particle trajectories on the x-y plane are shown in Fig. 1 by dotted curves. It is clear that every particle will just touch the solenoid axis and will turn back as is shown. In Fig. 1 we have shown the cross section of the periphery of the beam when it enters region II by a solid line, and four particles located at points A, B, C, and D on its periphery. As the beam travels through region II, the projection of the trajectories of these particles on the x-y plane is shown by dotted circles. After the beam travels a certain distance in region II, these particles move from locations A, B, C, and D to locations A′, B′, C′, and D′, respectively, as shown in Fig. 1. For clarity, we also have shown a particle located at E, which is inside the periphery of the beam. The trajectory of this particle is obtained by joining this point with the solenoid axis and then drawing a circle of diameter OE through these points. The particle moves to a location E′ when the other particles move to A′, B′, ... It is seen that the radius of the electron beam shrinks from OA to OA′, and the periphery of the beam shrinks to A′B′C′D′, as shown by the dashed curve in Fig. 1. In this manner, the beam undergoes periodic focusing in the region of uniform magnetic field in the solenoid.

As expected, the particles rotate with the cyclotron frequency about an axis passing through the center of their individual circular trajectories. The angular velocity about the axis of the solenoid is given by half of the cyclotron frequency because the angle they subtend at the axis of the solenoid is half of the angle that they subtend at the center of their individual circular trajectories (see Fig. 2). We clarify the confusion between the Larmour frequency and the cyclotron frequency using the simple geometry in Fig. 2. The particles rotate with the cyclotron frequency around the center of their individual trajectories but rotate with the Larmour frequency around the axis of the solenoid.

We now look at the trajectories of the single particles in more detail. As discussed, the angular velocity of the particle about the solenoid axis will remain constant and is given by
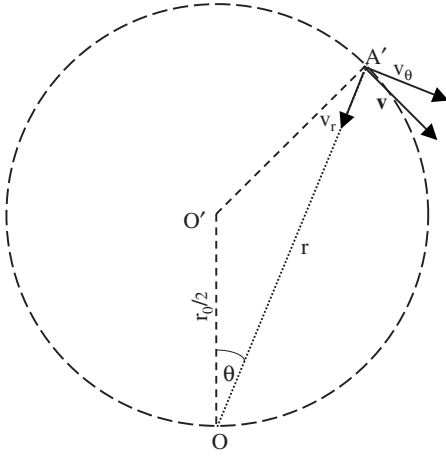
Fig. 3. Decomposition of the particle velocity into radial and azimuthal components with respect to the solenoid axis passing through point O.

ω_L. The magnitude of the transverse component $v_\perp$ of the velocity also remains constant at its initial value in region II at $z=0$, which is same as the magnitude of Δv_θ given by Eq. (5) because $v_r=0$ at $z=0$. The radial and azimuthal coordinates of the particle are given by

$$r = r_0 \cos\left(\frac{\omega_L z}{v_z}\right), \tag{7}$$

$$\theta = \theta_0 + \frac{\omega_L z}{v_z}, \tag{8}$$

where θ_0 is the initial value of θ and is zero for the case shown in Fig. 2. Note that the coordinates (r, θ) are measured relative to point O on the axis of the solenoid. Here, $z=v_z t$ is taken as the independent variable instead of the time t.

The particle velocity at an arbitrary location in region II can be decomposed into radial and azimuthal components, as shown in Fig. 3. The components are given by

$$v_r = -r\omega_L \tan\left(\frac{\omega_L z}{v_z}\right), \tag{9}$$

$$v_\theta = r\omega_L. \tag{10}$$

Equations (7) and (8) have been used to derive Eqs. (9) and (10).

We now look at what happens when the particle exits region II and enters region III. The particle again sees the B_r field, which is a Dirac delta function, and the Lorentz force due to coupling between B_r and v_z gives an impulse in the azimuthal direction. Consequently, v_θ undergoes a sudden change at this boundary given by

$$\Delta v_\theta = -r_1 \omega_L, \tag{11}$$

where r_1 is the radial coordinate of the particle at the exit of region II. By using Eqs. (10) and (11), we find the important result that at the entrance of region III, $v_\theta=0$. The radial velocity remains unchanged at the boundary and is given by

$$v_r = -r_1 \omega_L \tan\left(\frac{\omega_L L}{v_z}\right). \tag{12}$$

We assume that v_z does not undergo any change at the boundary due to the paraxial approximation. In region III, the particle moves in a straight line with constant v_r given by Eq. (12). After the beam undergoes periodic focusing in region II, the particles attain a radial velocity toward the solenoid axis [provided the tan term in Eq. (12) is positive], which is proportional to their radial displacement from the solenoid axis at the exit of region II, that is, $v_r=-kr_1$, where k is a constant. It can be easily shown that in this situation the beam will ideally be focused to a point after traveling a distance $z=v_z/k$ in region III. Thus the charged particle beam is focused in region III after passing through the solenoid.

III. SOME SPECIAL CASES

We first consider the case of a thin lens, that is, $L \ll v_z/\omega_L$. In this case, when the particle exits region II, the solenoid imparts a radial velocity to the particle given by

$$v_r = -\frac{r_0 e^2}{4\gamma^2 m^2 v_z} B_0^2 L, \tag{13}$$

where we have used the approximation that $\omega_L L/v_z \ll 1$. Equation (13) can be further generalized to a case where B_z has an arbitrary spatial variation if we assume the thin lens approximation so that the solenoid just gives an impulse to the electron and does not perturb its radial coordinate significantly. To do so, we assume that the form of B_z is given by Eqs. (3) and (4) piecewise, where B_0 varies from one segment to the other. In this way, Eq. (13) is generalized to the following form:

$$r' = -r\frac{e^2}{4\gamma^2 m^2 v_z^2} \int B^2 dz. \tag{14}$$

Here, $r'=dr/dz$ denotes the slope of the particle's trajectory. We use Eq. (14) to derive the formula for the focal length f of the thin solenoid lens as

$$\frac{1}{f} = \frac{e^2}{4\gamma^2 m^2 v_z^2} \int B^2 dz, \tag{15}$$

which is derived in many textbooks.[3,4] The integration is performed over the entire length of the solenoid.

We next discuss what happens when the electrons are generated inside a magnetic field, that is, when there is a residual magnetic field at the cathode and the electron beam comes out of the region of the magnetic field. This geometry corresponds to region I being absent and the cathode placed at $z=0$. We want to analyze what happens when the beam enters region III. The trajectory will be a straight line in region II because we are assuming a cold beam with no transverse component of velocity. When the beam exits region II, it will acquire an azimuthal velocity $v_\theta=-r\omega_L$. In this region particles will travel in a straight line. The trajectory of particles at locations P, Q, R, and S on the periphery of the beam are shown in Fig. 4. These particles move to the locations P′, Q′, R′, and S′, respectively, as is shown. The beam acquires an angular momentum and rotates and simultaneously expands as it propagates in the field-free region (see Fig. 4). As
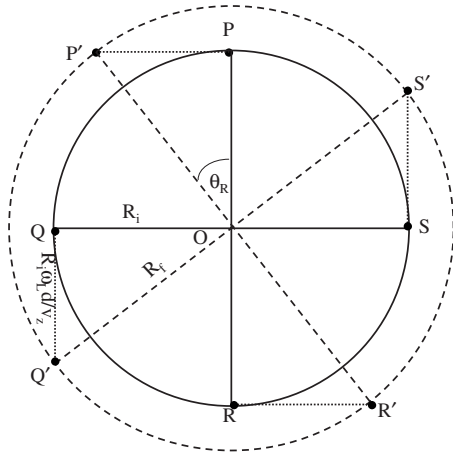
Fig. 4. Expansion and rotation of a beam with angular momentum. The solid curve is the periphery of the beam at the exit of region II, and the dashed curve is the periphery of the beam after it travels a distance d. The beam rotates by an angle θ_R and expands self-similarly.
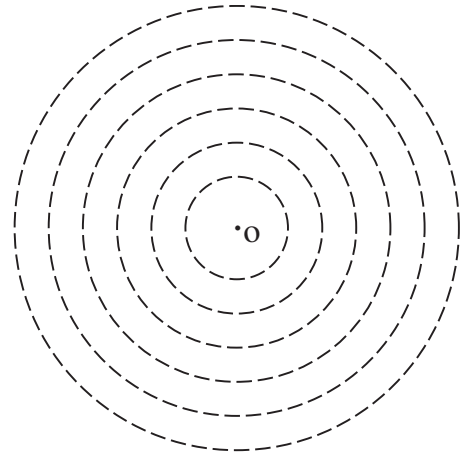

Fig. 5. Trajectories of electrons due to the solenoid magnetic field and the self-field of the electron beam when the electron beam radius is given by Eq. (20). The electron beam distribution is assumed to be uniform.

it expands, its angular velocity decreases to conserve angular momentum. The angular velocity of the beam after it travels a distance d in this region is given by

$$\omega = \omega_L \frac{v_z^2}{v_z^2 + \omega_L^2 d^2}. \tag{16}$$

The beam size increases as the beam propagates such that the angular momentum of the beam is conserved. The beam size R_f after the beam travels a distance d from the exit of region II is given by

$$R_f^2 = R_i^2 + \left(\frac{\omega_L R_i}{v_z}\right)^2 d^2, \tag{17}$$

which can be derived from Fig. 4. Here, R_i is the beam size at the exit of region II. The far-field beam divergence is therefore given by $\theta_d = \omega_L R_i / v_z$. The product of beam size at the beam waist and the far-field divergence is known as the emittance. The angular momentum of the beam thus gives rise to an equivalent emittance $\mathcal{E}$ for the beam given by

$$\mathcal{E} = \frac{eB_0 R_i^2}{2\gamma m v_z}, \tag{18}$$

which is known as the Busch emittance and is derived in Ref. 3 using the envelope equation. Our derivation implies that if there is a residual magnetic field at the cathode of the electron gun, the beam acquires an angular momentum and develops an equivalent emittance given by Eq. (18). In applications where low electron beam emittance is required, we need to make sure that the residual magnetic field at the cathode location is as small as possible. In some applications, we use this angular momentum to generate a flat electron beam with very low emittance in the vertical direction compared to the horizontal direction. To do so, we deliberately put a magnetic field at the cathode location to generate a beam with an angular momentum. A set of skew quadrupoles is then used to remove the angular momentum, generating a flat beam.[6]

We next discuss the effect of the space-charge force due to the self-field of the electron beam. The electron beam will generate its own electric and magnetic fields, which will af-fect its trajectory. These forces will be overall repulsive and will counteract the focusing effect due to the solenoid. The most interesting situation arises when the net force is such that the electrons rotate inside the solenoid in helical trajectories with radii equal to their radial distance from the solenoid axis. In this case, the entire beam would rotate around the solenoid axis with the Larmour frequency and with a constant beam radius, as shown in Fig. 5. Let us find out when this scenario can occur. If we assume the electron beam distribution to be uniform in a cylinder with constant radius and infinite length, the electric field and magnetic field can be calculated and the result for the radial force on an electron at a radial distance r from the solenoid axis is

$$F_r = 2\frac{mc^2}{\beta_z \gamma^2} \frac{I}{I_A} \frac{r}{R^2}. \tag{19}$$

Here, $\beta_z = v_z / c$, where c is the speed of light, I is the electron beam current, $I_A = 4\pi\epsilon_0 mc^3 / e = 17.04$ kA is the Alfvén current, ϵ_0 is the permittivity of free space, and R is the radius of the electron beam. This force will oppose the Lorentz force due to the solenoid field, which is $-e\omega_L B_z r$. It can be shown that if the magnitude of the space-charge force F_r becomes half of the magnitude of the Lorentz force due to the solenoid field, then the radius of the electron's trajectory becomes equal to its radial distance r from the solenoid axis. The requirement that the magnitude of the space-charge force becomes half of the magnitude of the Lorentz force leads to the condition

$$R = \sqrt{\frac{8m^2 c^2}{\gamma \beta_z e^2 B^2} \frac{I}{I_A}}. \tag{20}$$

If at the entrance of the solenoid the electron beam has a radius equal to the matched beam radius given by Eq. (20), the electrons will perform uniform circular motion at the Larmour frequency, with the center of the trajectories on the solenoid axis. For a given beam radius, the magnetic field can be chosen such that Eq. (20) is satisfied. In that case, the defocusing force due to self-fields will cancel the focusing force due to solenoid field, and the electron beam will maintain a constant radius in the solenoid. Note that we have not considered the diamagnetic field generated by the electron beam in the longitudinal direction opposing the solenoid

field, due to its azimuthal motion.[7] This effect is small and is neglected here.

IV. DISCUSSION

We have described the beam dynamics of a charged particle beam in an axisymmetric magnetic field of a solenoid and have clarified several issues regarding the focusing of charged particle beams in a solenoid magnetic field. It has been shown that a charged particle rotates with the cyclotron frequency about the axis of its helical trajectory in the uniform field region of the solenoid when the force due to beam self-fields are ignored. For different values of the radial coordinates of electrons at the entrance to the region of uniform magnetic field, we find that the angular velocity, even about the solenoid axis, is constant and is equal to half of the cyclotron frequency, which is the Larmour frequency. The trajectory of the electron on the transverse plane is obtained by drawing a circle of diameter equal to its radial distance from the solenoid axis, which passes through the solenoid axis and the electron's initial position. The analysis of the trajectories explains how the electron beam is periodically focused inside the uniform field region of the solenoid. The effect of the self-force of the beam was also discussed, and it was shown that if the electron beam radius is equal to the matched radius given by Eq. (20), the electrons rotate in circles centered on the solenoid axis with the Larmour frequency.

When the particle exits the solenoid, its azimuthal velocity becomes zero, but it has a radial velocity proportional to the radial distance of the particle from the solenoid axis. As a result, the beam is focused, and an expression was derived for the focal length of the solenoid. We also discussed the situation when there is a residual magnetic field at the cathode and the electrons emitted there come to a field-free region. The electron beam acquires an angular momentum in this case, which gives rise to a rotation and simultaneous self-similar expansion of the beam. The beam develops an equivalent emittance.

Our analysis can be easily extended to a beam with elliptical cross section. Inside the solenoid the beam will rotate and shrink in a self-similar fashion. Similarly, after the beam exits the solenoid with an angular momentum, it will rotate and expand in a self-similar fashion.

We ignored the effect of a random component of transverse velocities of electrons arising due to finite emittance. This effect will change the radius as well as the center of the electron's circular trajectory in Fig. 1. The change will be different for different electrons and will give an overall defocusing effect. A rigorous analysis of beam dynamics of a charged particle beam in the presence of a space-charge force and finite emittance can be done using the envelope equation.[3]

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