Angular Consistency of the Polarization Transfer of Electromagnetic Processes

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PEPPo TN-21-01, November 29, 2021

Abstract

The present note investigates the consistency of the description of unpolarized cross sections and polarization transfers of the bremsstrahlung and pair creation reactions. It shows the existence of an ambiguity in the early caluclations of Olsen & Maximon, with respect to the definition of an angular variable which is alternatively defined from the component of the electron momentum perpendicular to the photon momentum, or the component of the incident primary particle momentum perpendicular to the secondary particle momentum. The first definition supports consistent unpolarized cross section between different authors, while the second definition applies for polarization transfers. This ambiguity has no impact on the bremsstrahlung reaction but strongly affects the pair creation process.

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1 Introduction

The transfer of the polarization of incident particles in the bremsstrahlung and pair creation reactions are the basic mechanisms of the production of polarized positron beams through the PEPPo technique [1]. An accurate description of the polarization transfers is therefore mandatory for an optimized evaluation and implementation of a PEPPo-based polarized positron source.

Previous studies [2] reported some unphysical behaviour of the widely used calculations of Olsen & Maximon [4], and further proposed an empirical solution to resolve this issue [3]. The origin of this problem was ultimately identified by Kuraev *et al.* [5] as an effect of the finite mass of the electron in polarized cross sections, particularly significant at the boundary regions of the kinematical phase space, that are regions where one of the final state massive particle is close to rest.

This note revisits the derivation of the unpolarized cross sections of the bremsstrahlung and pair creation processes and investigates the consistency between different approaches developed in the ultra-relativistic and small angle limits. The findings of this study are then propagated in the early calculations of the polarization transfers to reveal further ambiguities beyond finite electron mass effects.

2 The Tsai cross sections

The theoretical investigation of the bremsstrahlung and pair creation processes has a long history of contributors starting from the early thirties [6]. For its completeness and excellent documentation, the work of Tsai [7] is selected here as a reference. It offers a modern discussion of these processes and a time-intermediate between Olsen & Maximon calculations and the most recent investigations [5, 8]. Following a inverted path with respect to Olsen & Maximon, *i.e.* starting from the pair creation reaction, this work also provides further handle on the kinematical substitution recipe leading to the expression of the bremsstrahlung cross section.

2.1 Pair creation

The pair creation cross section, as determined in Ref. [7], is defined as a 3-fold differential cross section which expresses as

$$\frac{d^3\sigma}{d^2\Omega\, dp} = \frac{2\alpha^3}{\pi}\,\frac{1}{k}\,\frac{E^2}{m^4}\,\left\{A_{xl}\,G_2(\infty) + B_{xl}\left(X - 2Z^2 f\left[(\alpha Z)^2\right]\right)\right\} \tag{1}$$

with

$$A_{xl} = \frac{2x(1-x)}{(1+l)^2} - \frac{12lx(1-x)}{(1+l)^4} \tag{2}$$

$$B_{xl} = \frac{2x^2-2x+1}{(1+l)^2} + \frac{4lx(1-x)}{(1+l)^4}\,, \tag{3}$$

and

$$x = E/k \tag{4}$$

$$l = E^2\theta^2/m^2\,. \tag{5}$$

The function $f\left[(\alpha Z)^2\right]$, which represents the Coulomb correction to the one-photon exchange approximation, is expressed as [9]

$$f(z) = z\sum_{n=1}^{\infty}\frac{1}{n(n^2+z)}\,. \tag{6}$$

$(k,\vec{k})$ represents the incoming photon, and $(E,\vec{p})$ is the electron emitted at an angle θ with respect to the photon†. In the ultra-relativistic and small angle limit, that is the hypothesis of these calculations, l represents the squared of the electron momentum component perpendicular to the photon momentum. Furthermore,

$$G_2(\infty) = Z^2 + Z \tag{7}$$

$$X = \int_{t_{min}}^{t_{max}}\left[\frac{a^4Z^2}{(1+a^2t^2)^2} + \frac{b^4Z}{(1+b^2t^2)^2}\right](t-t_{min})\,dt \tag{8}$$

with

$$a = \frac{184.15}{\sqrt{2.718}}\frac{m}{Z^{1/3}} \tag{9}$$

$$b = \frac{1194.}{\sqrt{2.718}}\frac{m}{Z^{2/3}}\,, \tag{10}$$

†Note the positive sign between the two terms of the right hand-side of Eq. (3), further confirmed in the erratum of Ref. [7].

and where

$$t_{min} = \left[\frac{m^2\,(1+l)^2}{2kx(1-x)}\right]^2 \quad (11)$$

$$t_{max} = m^2\,(1+l)^2\,. \quad (12)$$

2.2 Bremsstrahlung

The bremsstrahlung cross section is related to the pair creation cross section by the kinematical subtsitutions $k \leftrightarrow -k$ and $p \leftrightarrow -p$ in Eq. (1), were p represents the initial incoming electron and k is the emitted photon. The energy angle distribution of the bremsstrahlung can then be written as

$$\frac{d^3\sigma}{d^2\Omega_k\,dk} = -\left[\frac{d^3\sigma}{d^2\Omega\,dp}\right]_{\substack{k\leftrightarrow -k\\ p\leftrightarrow -p}} \frac{k^2 E}{p^3}\,. \quad (13)$$

Therefore

$$\frac{d^3\sigma}{d^2\Omega_k\,dk} = -\frac{2\alpha^3}{\pi}\,\frac{1}{(-k)}\,\frac{E^2}{m^4}\,\frac{k^2E}{p^3}\,\left\{A_{xl}\,G_2(\infty) + B_{xl}\left(X - 2Z^2 f\left[(\alpha Z)^2\right]\right)\right\}\,. \quad (14)$$

Introducing $y{=}k/E{=}1/x$, the (A_{xl}, B_{xl}) coefficients can be expressed as

$$A_{xl} = \frac{E^2}{k^2}\,A_{yl} = \frac{E^2}{k^2}\left[\frac{2(y-1)}{(1+l)^2} - \frac{12l(y-1)}{(1+l)^4}\right] \quad (15)$$

$$B_{xl} = \frac{E^2}{k^2}\,B_{yl} = \frac{E^2}{k^2}\left[\frac{2-2y+y^2}{(1+l)^2} + \frac{4l(y-1)}{(1+l)^4}\right] \quad (16)$$

such that

$$\frac{d^3\sigma}{d^2\Omega_k\,dk} = \frac{2\alpha^3}{\pi}\,\frac{1}{k}\,\frac{E^2}{m^4}\,\frac{E^3}{p^3}\left\{A_{yl}\,G_2(\infty) + B_{yl}\left(X - 2Z^2 f\left[(\alpha Z)^2\right]\right)\right\} \quad (17)$$

which, in the ultra-relativistic limit, corresponds to the expression (3.80) of Ref. [7]. Keeping the same physics meaning of related to the component of the electron momentum perpendicular to the photon momentum, the angular variable l is defined in the small angle limit as

$$l = E^2\theta_k^2/m^2 \quad (18)$$

and the minimum four-momentum transfer to the target becomes

$$t_{min} = \left[\frac{m^2k\,(1+l)^2}{2E(k-E)}\right]^2\,. \quad (19)$$

3 The Olsen & Maximon cross sections

The intent pursued here is to derive the cross sections of Ref. [4] from the previous more complete expressions presented in the previous section. For instance, the inelastic contributions scaling with the Z number of atomic electrons are not considered in Ref. [4], such that

$$G_2(\infty) = Z^2 \quad (20)$$

$$X = \int_{t_{min}}^{t_{max}} \frac{a^4 Z^2}{(1+a^2t^2)^2}\,(t - t_{min})\,dt \quad (21)$$

$$= Z^2\left(3 + 2\Gamma + 2f\left[(\alpha Z)^2\right]\right) \quad (22)$$

where the last equality is established in Ref. [7]. The Γ term comprises Coulomb and screening effects, and is written in Eq. (6.29) of Ref. [4] as

$$\Gamma = \mathcal{F}(\delta/\xi) - \ln(\delta) - 2 - f\left[(\alpha Z)^2\right] \tag{23}$$

where $\mathcal{F}(\delta/\xi)$ represents the screening effects defined in Eq. (6.31) of Ref. [4], δ is a kinematical variable, and ξ is an angular related variable.

3.1 Bremsstrahlung

Considering the bremsstrahlung process, we have from Eq. (17)

$$\frac{d^3\sigma}{d^2\Omega_k\, dk} = \frac{2Z^2\alpha^3}{\pi}\,\frac{1}{k}\,\frac{E^2}{m^4}\,\{A_{yl} + B_{yl}(3+2\Gamma)\}\ . \tag{24}$$

Following the notation changes where $(\varepsilon_1, \vec{p}_1)$ represents the incoming electron and $(\varepsilon_2, \vec{p}_2)$ is the scattered one, we have

$$y = \frac{k}{E} \equiv \frac{k}{\varepsilon_1} \tag{25}$$

$$1 - y = \frac{E-k}{E} \equiv \frac{\varepsilon_2}{\varepsilon_1}\,, \tag{26}$$

where $\varepsilon_1{=}k + \varepsilon_2$ defines the energy conservation law of the bremsstrahlung reaction, *i.e.* the kinetic energy of the recoiling nucleus is neglected. Additionnaly, energies are expressed in units of the electron mass, such that

$$l \quad = \quad E^2\theta_k^2/m^2 \equiv p_1^2\theta_1^2 = u^2 \tag{27}$$

$$dk \quad \equiv \quad m\, dk \tag{28}$$

Furthermore, the angular variable ξ is defined according to

$$\xi = 1/(1+u^2)\,, \tag{29}$$

such that

$$\theta_1 d\theta_1 = d\xi/(2p_1^2\xi^2)\,. \tag{30}$$

Integrating over the out-of-plane angle ϕ_1, we obtain

$$\frac{d^2\sigma}{d\xi\, dk} = \frac{2Z^2\alpha^3}{m^2}\,\frac{\varepsilon_1^2}{k}\,\frac{1}{p_1^2\xi^2}\,\{A_{yl} + B_{yl}(3+2\Gamma)\} \tag{31}$$

with

$$A_{yl} \quad = \quad -2\xi^2\frac{\varepsilon_2}{\varepsilon_1} + 12\xi^4u^2\frac{\varepsilon_2}{\varepsilon_1} \tag{32}$$

$$B_{yl} \quad = \quad \xi^2\left(1 + \frac{\varepsilon_2^2}{\varepsilon_1^2}\right) - 4\xi^4u^2\frac{\varepsilon_2}{\varepsilon_1}\,, \tag{33}$$

which yields

$$\frac{d^2\sigma}{d\xi\, dk} = \frac{2Z^2\alpha^3}{m^2}\,\frac{1}{k\,p_1^2}\,\left\{(\varepsilon_1^2+\varepsilon_2^2)(3+2\Gamma) - 2\varepsilon_1\varepsilon_2(1+4u^2\xi^2\Gamma)\right\}\ . \tag{34}$$

This strictly corresponds to Eq. (7.2) of Ref. [4]. Note that Eq. (30) is better replaced with

$$\sin(\theta_1)d\theta_1 = d\xi/2p_1\xi^{3/2}\sqrt{\xi(1+p_1^2)-1} \tag{35}$$

in the most general angular case.

3.2 Pair creation

Starting from Eq. (1) and following the same notation changes as previsouly, we have

$$\frac{d^3\sigma}{d^2\Omega_1\,dp_1} = \frac{2Z^2\alpha^3}{\pi m^2}\,\frac{\varepsilon_1^2}{k}\,\{A_{xl} + B_{xl}(3+2\Gamma)\} \tag{36}$$

with

$$A_{xl} = 2\frac{\xi^2}{k^2}\frac{\varepsilon_2}{\varepsilon_1} - 12\frac{\xi^4}{k^2}u^2\frac{\varepsilon_2}{\varepsilon_1} \tag{37}$$

$$B_{xl} = \frac{\xi^2}{k^2}\left(\varepsilon_1^2+\varepsilon_2^2\right) + 4\frac{\xi^4}{k^2}u^2\frac{\varepsilon_2}{\varepsilon_1}\,. \tag{38}$$

Integrating over the out-of-plane angle and changing the angular variable together with the elementary volume to $d\xi\,d\varepsilon_1$, the numerical coefficient in Eq. (36) becomes

$$2\pi\,\frac{2Z^2\alpha^3}{\pi m^3}\,\frac{\varepsilon_1^2}{k}\,\frac{1}{2\xi^2p_1^2}\,\frac{\varepsilon_1}{p_1} \tag{39}$$

leading, in the ultra-relativistic limit, to

$$\frac{d^2\sigma}{d\xi\,d\varepsilon_1} = \frac{2Z^2\alpha^3}{m^2}\,\frac{1}{k^3}\,\left\{(\varepsilon_1^2+\varepsilon_2^2)(3+2\Gamma) + 2\varepsilon_1\varepsilon_2(1+4u^2\xi^2\Gamma)\right\}\,. \tag{40}$$

This, once again, strictly corresponds to Eq. (10.4) of Ref. [4].

These comparisons establish univoquely that the angular variable which enters the unpolarized cross section for both the bremsstrahlung and pair creation processes is linked to the component of the electron momentum perpendicular to the photon. The consistency of Tsai and Olsen & Maximon expressions is further supported by the work of Ref. [10] which proposes an unified description of polarization phenomena in electromagnetic processes.

4 Polarization transfer

Considering specifically the pair creation case, the circular polarization (P_γ) of a photon beam transfers into longitudinal (P_z) and transverse (P_x) polarization components to the electron (or positron) following the expressions[†] [4, 10]

$$P_x = P_\gamma\,\frac{-\,4k\varepsilon_2 u\xi(1-2\xi)\Gamma}{(\varepsilon_1^2+\varepsilon_2^2)(3+2\Gamma)+2\varepsilon_1\varepsilon_2(1+4u^2\xi^2\Gamma)} \tag{41}$$

$$P_z = P_\gamma\,\frac{k\,\left[(\varepsilon_1-\varepsilon_2)(3+2\Gamma)+2\varepsilon_2(1+4u^2\xi^2\Gamma)\right]}{(\varepsilon_1^2+\varepsilon_2^2)(3+2\Gamma)+2\varepsilon_1\varepsilon_2(1+4u^2\xi^2\Gamma)}\,. \tag{42}$$

For linearly polarized photons, the pair is most likely emitted in the plane of polarization with an in-plane/out-of-plane asymmetry determined by the quantity

$$A_{\parallel/\perp} = \frac{8\varepsilon_1\varepsilon_2u^2\xi^2\Gamma}{(\varepsilon_1^2+\varepsilon_2^2)(3+2\Gamma)+2\varepsilon_1\varepsilon_2(1+4u^2\xi^2\Gamma)}\,. \tag{43}$$

[†]Note the negative sign in Eq. (41), needed for consistency with Fig. 8 of Ref. [4].

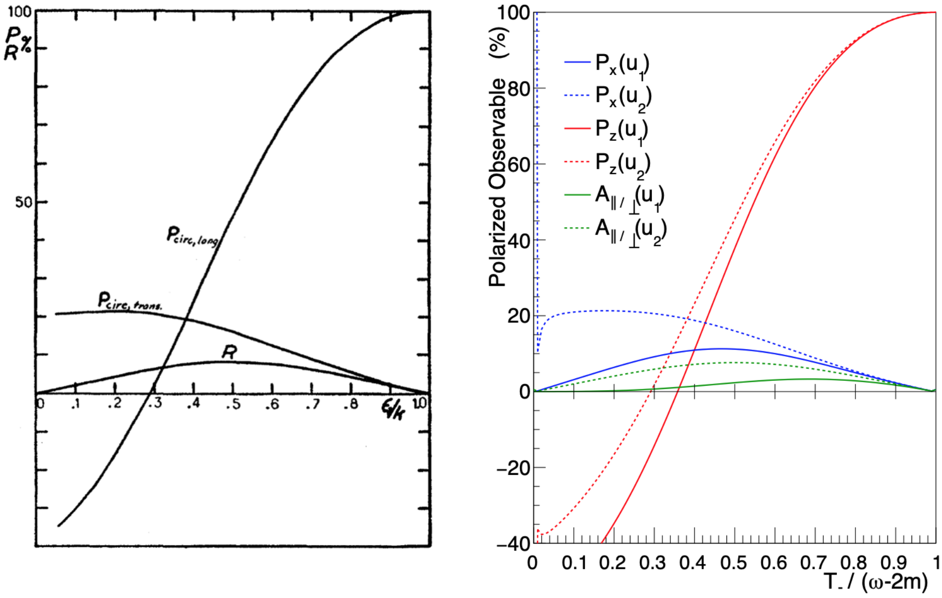

Figure 1: (left) Fig. 8 of Ref. [4] showing the energy distribution of $P_{circ,trans.} \equiv P_x$, $P_{circ,long} \equiv P_z$, and $R \equiv A_{\parallel/\perp}$, for electrons produced off lead at an angle of 0.41 mrad from circularly polarized photons of 500 MeV. (right) The same quantities determined using the u_1 (solid line) or u_2 (dashed line) definition of the angular variable.

Following the previous sections, the angular variable entering these expressions should correspond to the electron momentum component perpendicular to the photon momentum, that is

$$u \equiv u_1 = p_1 \sin(\theta_1)\,. \tag{44}$$

For the pair creation process, this definition differs from the conjecture of the component of the primary incident particle perpendicular to the secondary particle momentum, which writes

$$u \equiv u_2 = k \sin(\theta_1)\,. \tag{45}$$

The polarization observables previously defined are shown in Fig. 1, determined from their respective expressions for a photon beam of 500 MeV and an electron emission angle of 0.41 mrad off a lead target. They are further compared to the Fig. 8 of Ref. [4] determined under the same conditions. It is readily seen that polarization observables determined using u_1 does not reproduce earlier calculations of Ref. [4]. Using u_2 appears to be the solution compatible with these calculations, then questionning the results of Ref. [4]. Is it a typo in the written expressions or an error in the production of the figure? In that respect, further theoretical work did not bring new insight, especially Ref. [10] which contains several typos. We note that the implementation of polarized pair creation [11] into GEANT4 [12] uses the angular variable u_1, and that the modern approach developed in Ref. [5] does not suffer from such ambiguity.

5 Conclusion

The present study supports the existence of an ambiguity in the definition of the angular variable for the pair creation process which appears to be different for cross sections and polarization observables within the same framework of Ref. [4]. This feature adds to the previously reported unphysical behaviour of the ultra-relativistic limit [2], and calls for an independent determination of polarization observables from first principles *i.e.* free from eventual angular ambiguities that may arise from the kinematical substitution recipe.

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