4.2. Stern-Gerlach beam deflection by externally-powered cavity. From Eq. (147), the transverse forces applied to a particle due to its magnetic moment are

$$R_x^m = \mu_x^* \frac{\partial B_x}{\partial x} + \mu_y^* \frac{\partial B_y}{\partial x}, \quad R_y^m = \mu_x^* \frac{\partial B_x}{\partial y} + \mu_y^* \frac{\partial B_y}{\partial y}. \tag{150}$$

Copying from Eqs. (61) through (63), and evaluating the only non-vanishing transverse derivative of a transverse component,

$$E_{0y} = -e^{-jk_0(z_0 \cos\alpha_0 + x_0 \sin\alpha_0)}\, e^{j\omega_0 t_0}\, E_0, \tag{151}$$
$$B_{0x} = \cos\alpha_0\, e^{-jk_0(z_0 \cos\alpha_0 + x_0 \sin\alpha_0)}\, e^{j\omega_0 t_0}\, B_0, \tag{152}$$
$$B_{0z} = -\sin\alpha_0\, e^{-jk_0(z_0 \cos\alpha_0 + x_0 \sin\alpha_0)}\, e^{j\omega_0 t_0}\, B_0, \tag{153}$$
$$\frac{\partial B_{0x}}{\partial x_0} = -jk_0 \sin\alpha_0 \cos\alpha_0\, e^{-jk_0(z_0 \cos\alpha_0 + x_0 \sin\alpha_0)}\, e^{j\omega_0 t_0}\, B_0. \tag{154}$$

In our impulse approximation, the particle is treated as stationary at the origin during the time interval that the cavity is passing. The momentum transfered from wave to particle is accumulated, but the effect of any recoil displacement during this time is neglected. We therefore set $x_0 = z_0 = 0$ and substitute into the first of Eqs. (150);

$$R_{0x}^{m_\perp TE} = \mu_x^* \frac{\partial B_{0x}}{\partial x_0} = -jk_0 \mu_x^* \sin\alpha_0 \cos\alpha_0\, e^{j\omega_0 t_0}\, B_0. \tag{155}$$

Following the same procedure as in Eqs. (71), we can calculate the maximum possible transverse momentum impulse that can be administered to the particle by a skew wave during one half cycle (which is the maximum possible);

$$\begin{aligned}
\Delta p_{0x}^{\perp\max} c &= \frac{c}{\omega_0} \int_0^\pi R_{0y}^{m_\perp TE}(t_0, 0)\, d(\omega_0 t_0) \\
&= -j\mu_x^* B_0 \sin\alpha_0 \cos\alpha_0 \int_0^\pi (\cos\omega_0 t_0 + j \sin\omega_0 t_0)\, d(\omega_0 t_0) \\
&= 2\mu_x^* B_0 \sin\alpha_0 \cos\alpha_0.
\end{aligned} \tag{156}$$

The sign here could be reversed by shifting the RF phase by π. To complete the transformation to laboratory parameters, noting from Figure 7 that $B_{0z} = -B_0 \sin\alpha_0$, and using the fact that $B_\parallel$ is invariant,

$$-B \sin\alpha = B_z = B_{0z} = -B_0 \sin\alpha_0. \tag{157}$$

can be used. (Note that B is the (transverse to propagation direction) laboratory frame magnetic field of the skew wave, *not* the total longitudinal magnetic field summed over skew waves.) Also (in fully-relativistic approximation)

$$\cos\alpha_{0\pm} = -\sqrt{\frac{1}{1 + \tan^2 \alpha_{0\pm}}}, \tag{158}$$

showing that both $\cos\alpha_{0\pm}$ values are negative (as has been explained previously). Substituting these expressions into Eq. (156) yields

$$\Delta p_{0x}^{\perp\max} c = 2\mu_x^*(-B\sin\alpha)\sqrt{\frac{1}{1+\tan^2\alpha_{0\pm}}} \approx 2\mu_x^*(-B\sin\alpha). \tag{159}$$

Finally we note that the transverse component of recoil momentum is unchanged by Lorentz transformation, and obtain

$$\Delta p_x^{\perp\max} c \approx 2\mu_x^*(-B\sin\alpha). \tag{160}$$

We retain the negative sign only for consistency, even though the sign of the momentum impulse can be reversed by reversing the RF phase, as noted in connection with Eq. (156).

Assuming fully relativistic kinematics, the laboratory longitudinal momentum of the particle is $p = \gamma m_e c$. As a result the laboratory deflection angle due to a single skew wave is given by

$$\Delta\theta_1 \approx \frac{2\mu_x^* B\sin\alpha}{\gamma m_e c^2} \approx \frac{\sqrt{2}\mu_B B}{\gamma m_e c^2}, \tag{161}$$

where approximation $\sin\alpha \approx 1/\sqrt{2}$ has been made and μ_x^* has been replaced by Bohr magnetron $\mu_B = 5.78\times 10^{-5}\,\mathrm{eV/T}$. For the TE_{101} mode all 4 skew waves interfere constructively and the total deflection is given by

$$\Delta\theta \approx \frac{2\mu_B B_\perp}{\gamma m_e c^2}, \tag{162}$$

This formula can be compared to a standard formula for non-relativistic Stern-Gerlach deflection, by a DC magnet of length L_z with magnetic field gradient $dB_\perp/dx$, for a spin 1/2 molecule traveling at speed V_z along the z-axis;

$$\Delta\theta_{\mathrm{NR}} \approx \frac{\mu_B L_z dB_\perp/dx}{MV_z^2}. \tag{163}$$

4.2.1. *Numerical example of Stern-Gerlach deflection.* We consider a case in which our TE_{101} rectangular cavity is driven at power $P_{ext} = 10^4\,\mathrm{W}$. According to Eqs. (9) and (13) the magnetic field is given (roughly) by

$$B_\perp = \mu_0 H_\perp = \mu_0\sqrt{\frac{Q_{\mathrm{rect.}}}{2\pi f_r}\,\frac{8}{abd\mu_0}\,\frac{1}{P^{\mathrm{ext}}}\,P^{\mathrm{ext}}}, \quad \mu_0\sqrt{[\mathrm{s}]\left[\frac{\mathrm{A}^2}{\mathrm{Jm}^2}\right]\left[\frac{\mathrm{s}}{\mathrm{J}}\right]}\ \left[\frac{\mathrm{J}}{\mathrm{s}}\right], \tag{164}$$

where numerical values of the various parameters are given by

$$
\begin{aligned}
\text{bunch frequency } f_0 &= 0.5 \times 10^9\,\text{Hz},\\
\text{beam current } I_e &= 100\,\mu\text{A},\\
\text{electrons/bunch } N_e &= 1.3 \times 10^6,\\
\text{electron energy } \mathcal{E}_e &= 1\,\text{MeV},\\
\text{relativistic gamma } \gamma_V &= 1.96,\\
\text{electron magnetic moment } \mu_e^* &= -0.928 \times 10^{-23},\ \text{J/T},\\
\text{Resonator Q} - \text{value } Q_r &= 29700,\\
\text{resonator frequency } f_r &= 0.75\,\text{GHz},\\
\text{resonator dimensions } a/b/d &= 0.292/0.146/0.274\,\text{m}\\
\frac{a\,b\,d\,\mu_0}{8} &= 1.83 \times 10^{-9}\,\frac{\text{J}}{(\text{A/m})^2},\\
\sqrt{\frac{Q_{\text{rect.}}}{2\pi f_r}\,\frac{8}{abd\mu_0}\,\frac{1}{P^{\text{ext}}}} &= \frac{58.6}{\sqrt{P^{\text{ext}}}},\\
\text{external power } P_{\text{ext}} &= 10^4,\ \text{W},\\
B_\perp &= 0.0074\,\text{T},\\
\text{beam deflection angle } \Delta\theta &= 0.85 \times 10^{-12}\,\text{radian},\\
\text{lattice functions } \beta_1/\beta_2 &= 50/100\,\text{m},\\
\text{betatron amplitude at } \beta_2\Delta x &= 0.6 \times 10^{-10}\,\text{m}.
\end{aligned}
\tag{165}
$$

The last few lines in the previous table give the Stern-Gerlach angular deflection $\Delta\theta$ caused by a cavity located at a point where the lattice beta function is β_1, and the displacement Δx at a downstream location with beta function β_2 and betatron phase $\pi/2$.

According to these calculations, to confirm the S-G deflection it will be necessary to detect betatron oscillation having Angstrom-scale amplitude. This should be possible, even with room temperature detection, since the drive frequency is externally controlled, and very narrow-band filtering over long sampling time can be employed.

Once the (non-controversial) transverse Stern-Gerlach effect has been confirmed, one will be able to address the Stern-Gerlach energy excitation with more confidence.