Penfold-Leiss Unfolding

Bubble Chamber Collaboration

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When using the continuous spectrum of Bremsstrahlung photon beam to study photo-nuclear reactions, the measured quantity is a yield. The yield (number of interactions) is a convolution of the cross section with the Bremsstrahlung spectrum:

$$y(E) = \int_{Threshold}^{E} N_{\gamma}(E,k)\sigma(k)dk, \tag{1}$$

where E is the electron beam kinetic energy, $N_{\gamma}(E,k)$ is the number of gammas per energy unit which depends on the electron energy and the gamma energy. The continuous range of photon energies means that the cross section is not measured directly, instead it must be unfolded from the measured yields.

An integral equation of this form is known as Volterra Integral Equation of the First Kind. Mathematically the problem is one of numerical solution of the yield integral equation and $\sigma(k)$ is the function to be solved for. One way to solve this equation is to use the Method of Quadratures (a method for constructing an approximate solution of an integral equation based on the replacement of integrals by finite sums). First the yields are measured at $E = E_1, E_2, \ldots, E_n$ where $E_i - E_{i-1} = \Delta$, $i = 2, \ldots, n$. Then,

$$y(E_i) = \int_{Threshold}^{E_i} N_{\gamma}(E_i,k)\sigma(k)dk \approx \sum_{j=1}^{i} N_{\gamma}(E_i,\Delta,k_j)\sigma(k_j), \tag{2}$$

where $N_{\gamma}(E_i,\Delta,k_j)$ is the number of gammas in the energy bin of width Δ.

Equation 2 is a set of linear equations which can be written in the matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} N_{11} & 0 & \cdots & 0 \\ N_{21} & N_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_{n1} & N_{n2} & \cdots & N_{nn} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix}. \tag{3}$$

This matrix equation can be solved with matrix inversion.
Equivalently, the solution to Equation 2 can be written:

$$\sigma_i = \frac{1}{N_{ii}} \left[y_i - \sum_{j=1}^{i-1} (N_{ij}\sigma_j) \right]. \tag{4}$$

The error propagation of Equation 4 is given by:

$$\left(\frac{\mathrm{d}\sigma_i}{\sigma_i}\right)^2 = \frac{\left[(\mathrm{d}y_i)^2 + \sum_{j=1}^{i-1}(N_{ij}\mathrm{d}\sigma_j)^2\right]}{\left[y_i - \sum_{j=1}^{i-1}(N_{ij}\sigma_j)\right]^2}. \tag{5}$$

For mono-chromatic photon beam, Equation 5 reduces to:

$$\left(\frac{\mathrm{d}\sigma_i}{\sigma_i}\right)^2 = \left(\frac{\mathrm{d}y_i}{y_i}\right)^2 = \frac{1}{y_i}. \tag{6}$$

Initially, the above unfolding method known as Penfold-Leiss unfolding ([1]) (aka the Inverse-Matrix Method) gave unreliable results (see for example [2] and Figure 1) because (in the sixties and seventies) the unfolding procedures have been often considered in isolation from the photon energy spectrum of the bremsstrahlung beam used experimentally. At that time, experimentalists used the Schiff theoretical formula ([3]) to calculate $N_{ij} = \Delta N_{\text{Schiff}}(E_i, k_j - \Delta/2)$. Findlay proposed ([4]) that a simple modification to N_{ij} prevents the generation of spurious results. He replaced $k - \Delta/2$ by $k - \lambda\Delta$ where λ is a parameter determined by considering the energy spread of the electron beam and the energy loss of the electron beam in the radiator. Findlay's modification was successfully demonstrated to produce correct cross sections in ([5], see Figure 2, [6]).

These days, there are very accurate Monte-Carlo simulations, N_{ij} can be calculated for each specific experimental conditions without the need to use theoretical formula. This removes problems in the unfolding related to the knowledge of N_{ij}.

However, this is not the only reason that may cause Penfold-Leiss unfolding to fail. Careful inspection of Equation 5 reveals that statistical errors

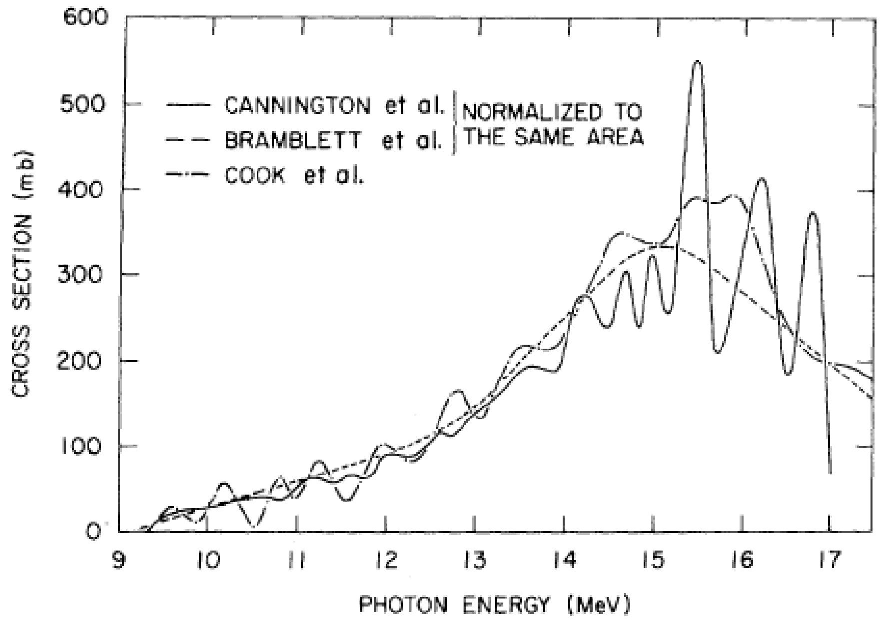

Figure 1: Comparison of the $^{141}\mathrm{Pr}(\gamma, n)$ cross sections. Cook and Cannington were done with Bremsstrahlung beams. The correct cross section by Bramblett (dashed line) was done with a monochromatic gamma beam from positron annihilation in flight.

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of the measured yields play a role in two ways. First, the statistical errors add up as can be seen in the numerator of the right hand side of Equation 5. Although σ_1 and probably σ_2 will be very closed to their real values, the remaining cross section data points will start to oscillate. Second, the denominator of the difference of two large numbers and thus will enhance the error in the cross section since the difference will be a smaller number. Indeed, having a very steep cross section is an advantage here, since it reduces the second term in the denominator and give a denominator with large number. To determine the required statistical error for each yield measurement, the steepness of the cross section must be taken into account. A relatively flat cross section requires very accurate yield measurements to be able to successfully unfold the cross section.

Indeed, the $^{16}\mathrm{O}(\gamma, \alpha)^{12}\mathrm{C}$ cross section is very steep (shown in Figure 3) and only photons near the endpoint contribute to the yield for each

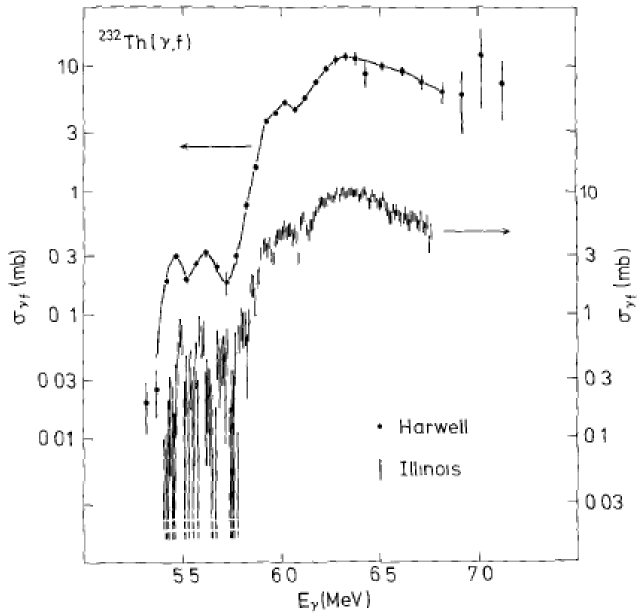

Figure 2: Photofission of ^{232}Th near threshold. Solid circles present data from [5] with Bremsstrahlung beam and lines data with tagged photon beam.

beam energy, see Figure 4. Similar arguments show that it is beneficial to maximize the number of gammas near the endpoint relative to the number of gammas at lower gamma energies, N_{ii}/N_{ij}, $j = 1, \ldots, i-1$. Figure 5 shows the Schiff Bremsstrahlung cross section for 8.5 MeV elctron beam kinetic energy. This is one reason, among many others, why we choose to run with a very thin Bremsstrahlung radiator. Figure 6 shows the Bremsstrahlung yield for three different radiator thicknesses.

As was discussed above, poor statistics will cause the unfolded cross section to oscillate as a function of photon energy especially at energies above the giant resonance energy where the cross section is flat or decreasing. These cross sections are unacceptable physically and smoothing must be used. Under the assumption of the cross section smoothness, the deconvolution method is known as the Regularization Method. There are several kinds of regularization methods such as Cook's Least Structure Method [7], the Second Difference Method [8] and Tikhonov Regularization [9].

Another deconvolution method is called the Photon Difference Method [10]. In this method a nearly mono-energetic photon spectrum can be constructed artificially by taking an algebraic sum of three Bremsstrahlung

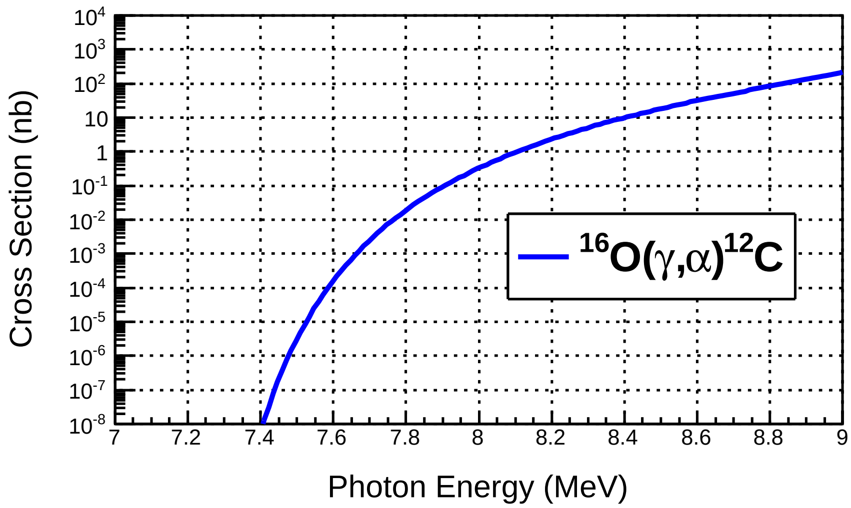

Figure 3: The cross section of $^{16}O(\gamma,\alpha)^{12}C$.

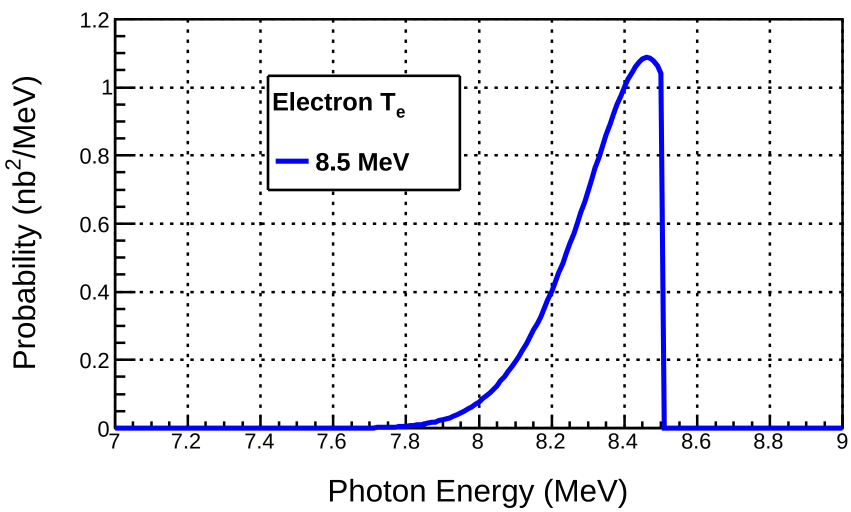

Figure 4: The probability of Bremsstrahlung photons to undergo the interaction $^{16}O(\gamma,\alpha)^{12}C$.

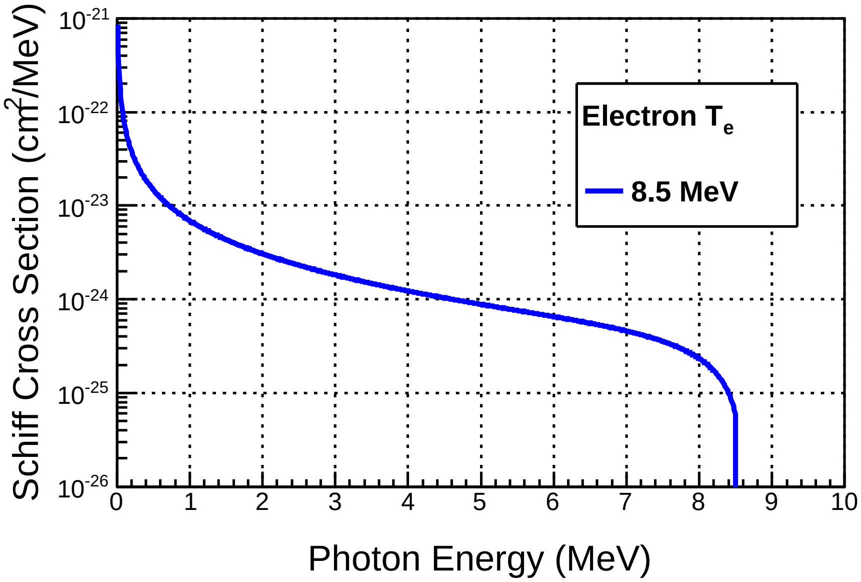

Figure 5: Schiff Bremsstrahlung cross section for a Cu radiator.

spectra with consecutive endpoint energies:

$$\phi_{\mathrm{M}}(E_i) = \phi_{\mathrm{Schiff}}(E_i) - a\ \phi_{\mathrm{Schiff}}(E_{i-1}) + b\ \phi_{\mathrm{Schiff}}(E_{i-2}), \tag{7}$$

where $\phi_{\mathrm{Schiff}}(E_i)$ is the Schiff Bremsstrahlung spectrum with endpoint energy E_i and the parameters a and b are both positive and chosen such that $\phi_{\mathrm{M}}(E_i)$ represent a mono-energetic photon spectrum. An example is shown in Figure 7 where:

$$\phi_{\mathrm{M}}(8.5) = \phi_{\mathrm{Schiff}}(8.5) - 1.35\ \phi_{\mathrm{Schiff}}(8.4) + 0.30\ \phi_{\mathrm{Schiff}}(8.3). \tag{8}$$

A differential yield spectrum can be constructed artificially by using the same linear combination of the corresponding yields. Then, the photonuclear cross section is simply the ratio of this differential yield to the corresponding mono-energetic photon flux.

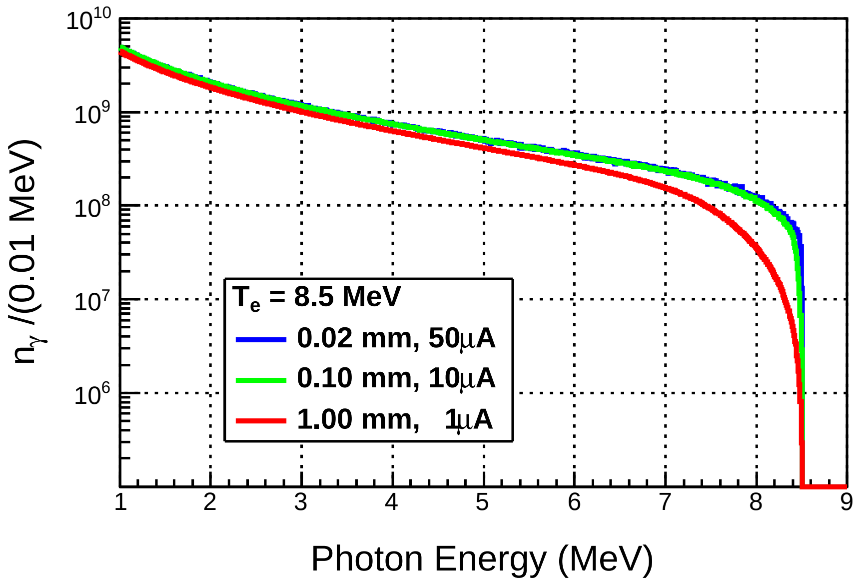

Figure 6: Bremsstrahlung yield for three different radiator thicknesses.

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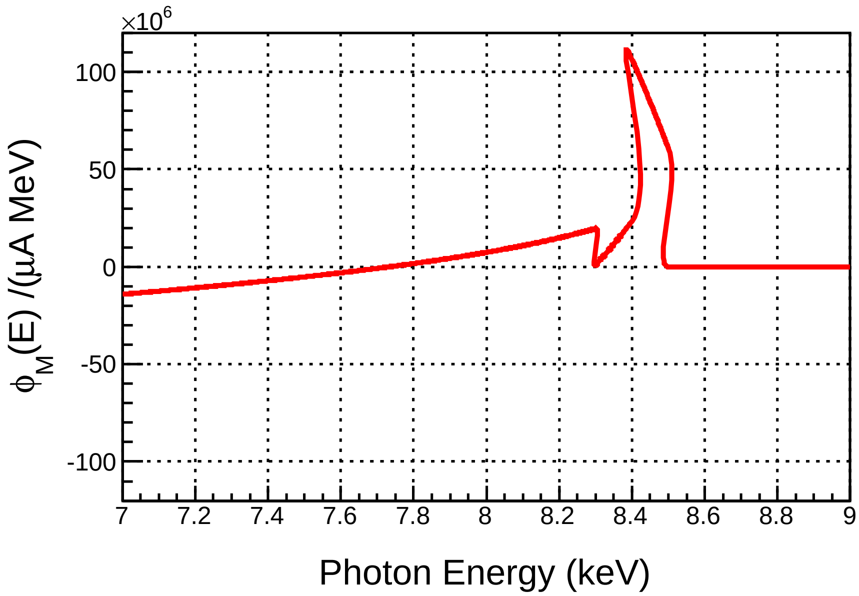

Figure 7: Mono-energetic photon spectrum constructed artificially by taking an algebraic sum of three Bremsstrahlung spectra with consecutive endpoint energies.

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