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Space Charge Benchmarking Derivation - a positron and an electron

1 Purpose

Benchmarking the spacecharge3D routine in GPT - a positron and electron 0.5 m apart travel in the z direction with initial velocities v_{0p} and v_{0e} resepectively. (We'll assume that $v_p, v_e \ll c$ to keep things non-relativistic). We'll check several cases for v_p and v_e and, using GPT simulations and theoretical calculations, calculate when ion cross paths. The ion will start at $(0, -0.25, 0)$ and the electron will start at $(0, 0.25, 0)$.

2 Theoretical Derivation

2.1 Lagrangian

The full Lagrangian for the system is

$$L = T_p + T_e - q_p\phi_p - q_e\phi_e - \frac{q_p}{c}\vec{A}_p \cdot \vec{v}_p - \frac{q_e}{c}\vec{A}_e \cdot \vec{v}_e \tag{1}$$

where T is the kinetic energy, q is the charge, ϕ is the electric potential, c is the speed of light and $\vec{A}$ is the vector potential. In the limit of $v \ll c$, the last two terms can be neglected, so we have

$$L = T_p + T_e - q_p\phi_p - q_e\phi_e \tag{2}$$

The electric potentials ϕ_p and ϕ_e are related by:

$$\phi_p = \frac{1}{4\pi\varepsilon_0}\frac{q_e}{r} \tag{3}$$

$$\phi_e = \frac{1}{4\pi\varepsilon_0}\frac{q_p}{r} = \frac{q_p}{q_e}\phi_p \tag{4}$$

where $\frac{1}{4\pi\varepsilon_0} \approx 9 \times 10^9\ \mathrm{N\,m^2\,C^{-2}}$ is the Coulomb constant and r is the distance away from the source charge q (in this case, r is the distance between the positron and electron). Using these definitions, we can rewrite (2) as

$$L = T_p + T_e - 2q_p\phi_p \tag{5}$$

$$L = \frac{1}{2}\left(m_p v_p^2 + m_e v_e^2\right) - \frac{1}{2\pi\varepsilon_0}\frac{e^2}{r} \tag{6}$$

where e is the elementary charge. Restricting our attention to the $yz-$plane, let $v^2 = \vec{v} \cdot \vec{v} = v_y^2 + v_z^2 = \dot{y}^2 + \dot{z}^2$ and $r = |\vec{r}| = \sqrt{(y_e - y_p)^2 + (z_e - z_p)^2}$, we can rewrite L as:

$$L = \frac{1}{2}\left[m_p\left(\dot{y}_p^2 + \dot{z}_p^2\right) + m_e\left(\dot{y}_e^2 + \dot{z}_e^2\right)\right] - \frac{e}{2\pi\varepsilon_0}\frac{1}{\sqrt{(y_e - y_p)^2 + (z_e - z_p)^2}} \tag{7}$$

2.2 Euler-Lagrange Equation

The equations of motion come from the Euler-Lagrange equation using r and $\dot{r}$ as the generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \tag{8}$$

which can be expanded into y and z as before:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z} \tag{10}$$

We'll restrict our attention to motion in the y direction, since the motion in the z direction is constant. We have two equations of motion: one for each particle:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_p}\right) = \frac{\partial L}{\partial y_p} \tag{11}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_e}\right) = \frac{\partial L}{\partial y_e} \tag{12}$$

For the positron, we have:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_p}\right) &= \frac{\partial L}{\partial y_p} \\ m_p \ddot{y}_p &= -\frac{e^2}{2\pi\varepsilon_0} \frac{y_e - y_p}{\left((y_e - y_p)^2 + (z_e - z_p)^2\right)^{\frac{3}{2}}}\end{aligned} \tag{13}$$

Similarly for the electron, we have:

$$\begin{aligned}m_e \ddot{y}_e &= -\frac{e^2}{2\pi\varepsilon_0} \frac{y_p - y_e}{\left((y_e - y_p)^2 + (z_e - z_p)^2\right)^{\frac{3}{2}}} \\ &= -m_p \ddot{y}_p\end{aligned} \tag{14}$$

as expected. In our case, the electron and positron have the same, constant velocity in the z, $z_e = z_p$ always, so we can neglect the second term in the denominator:

$$m_p \ddot{y}_p = -\frac{e^2}{2\pi\varepsilon_0} \frac{1}{|y_e - y_p|^2} \tag{15}$$

$$m_e \ddot{y}_e = \frac{e^2}{2\pi\varepsilon_0} \frac{1}{|y_e - y_p|^2} \tag{16}$$

2.3 Intercept location

Since the acceleration of both particles is equal and opposite in magnitude, they have the same mass, and they start the same distance away from the z axis, they should hit the z axis at the same time. Thus, we'll restrict our attention to the motion of the positron and determine the time it takes for the positron to hit the z axis. Once we know the time, we can figure out how far down the z axis the two particles will intercept based on the initial velocity in the z direction and compare this location with the simulation. Solving (15) for $\ddot{y}_p$:

$$\ddot{y}_p = \frac{\partial^2 y_p}{\partial t^2} = -\frac{e^2}{2\pi\varepsilon_0} \frac{1}{|y_e - y_p|^2}$$

Since we're concerned with the positron's distance from the z-axis, we can rewrite this equation using $y_p = y$ and $|y_e - y_p| = 2y$:

$$\begin{aligned}\ddot{y} &= -\frac{e^2}{4\pi\varepsilon_0} \frac{1}{y^2} \\ \frac{\partial^2 y}{\partial t^2} &= -\frac{C}{y^2}\end{aligned}$$

where $C = \dfrac{e^2}{4\pi\varepsilon_0}$. Multiplying both sides by $\dfrac{dy}{dt}$ and integrating yields:

$$\begin{aligned}
\int \frac{dy}{dt}\frac{d^2y}{dt^2}dx &= -C\int \frac{1}{y^2}\frac{dy}{dt}dt \\
\frac{1}{2}\left(\frac{dy}{dt}\right)^2 &= c_1 + \frac{C}{y} \\
\frac{dy}{dt} &= \pm\sqrt{2c_1 + \frac{2C}{y}} \qquad (17) \\
\frac{\frac{dy}{dt}}{\sqrt{2c_1 + \frac{2C}{y}}} &= \pm 1
\end{aligned}$$

Integrating with respect to t again yields:

$$\int \frac{\frac{dy}{dt}}{\sqrt{2c_1 + \frac{2C}{y}}} = c_2 \pm t$$

WolframAlpha gives for the integral

$$\left(\frac{y\sqrt{c_1 + \frac{2C}{y}}}{c_1} - \frac{C\log\left(\sqrt{c_1 + \frac{2C}{y}} + c_1 t + C\right)}{c_1^{\frac{3}{2}}}\right)^2 = (c_2 + t)^2 \qquad (18)$$

This is a transcendental equation (i.e. one cannot solve for $y(t)$ in this equation). Using $\frac{dy}{dt}(y = 0.25) = 0$ (i.e. the initial velocity is zero), we can solve for c_1 in (17):

$$\begin{aligned}
0 &= \pm\sqrt{2c_1 + \frac{2C}{0.25}} \\
0 &= 2c_1 + 8C \\
c_1 &= -4C \qquad (19)
\end{aligned}$$

Plugging this into(18) yields:

$$\left(-\frac{y\sqrt{\frac{2C}{y} - 4C}}{4C} - \frac{C\log\left(\sqrt{\frac{2C}{y} - 4C} + C(1 - 4t)\right)}{(-4C)^{\frac{3}{2}}}\right)^2 = (c_2 + t)^2$$

Clearly something is wrong, since we cannot have an imaginary number in the denominator...