

# INITIAL STATE PERSPECTIVE ON THE RIDGE

DILUTE-DENSE REGIME

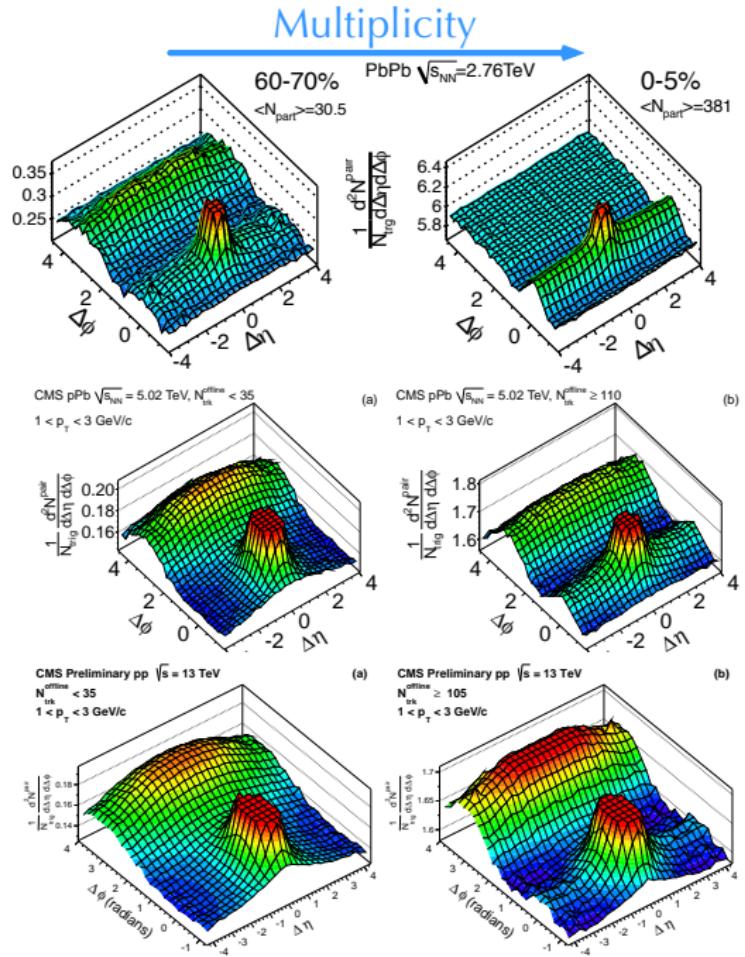
**Vladimir Skokov**



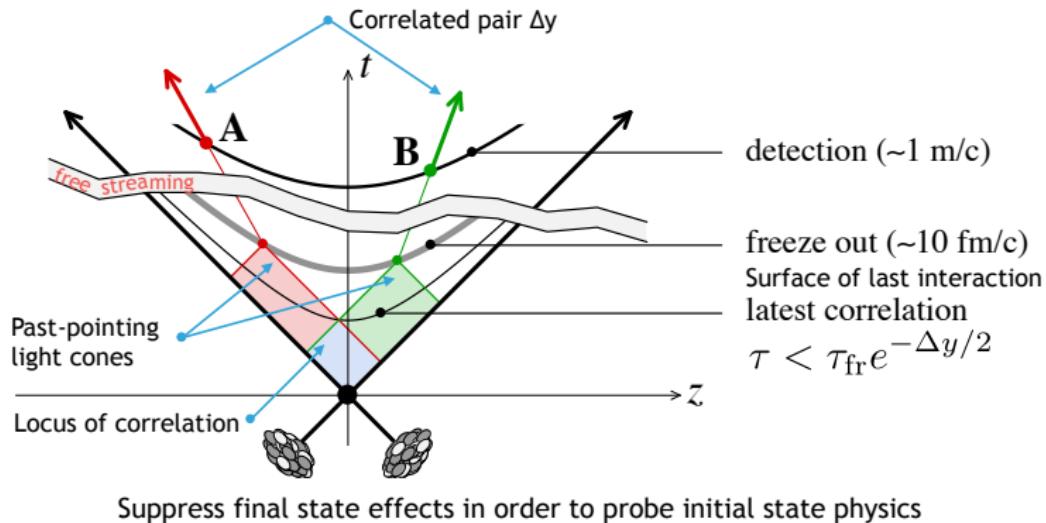
# Striking resemblance

- Present also in multi-particle correlations
- Collectivity in systems of drastically different sizes
- Does it imply the same underlying mechanism?

System Size ↑



# LONG-RANGE CORRELATIONS



- Regardless of nature of the ridge
  - long-range rapidity correlations either pre-exist in initial wave function or develop very early after collision
  - understanding initial/early stage is of paramount importance for understanding p-A and p-p.

figure adapted from A. Dumitru, F. Gelis, L. McLerran and R. Venugopalan, arXiv:0804.3858

# COLLECTIVITY: POSSIBLE ORIGIN

## Final state “hydro”

- Driven by initial state geometric correlations
- Develops gradually during (hydro) evolution
- Requires large multiplicities to facilitate final state interaction
- Requires non-trivial initial state geometry (proton shape fluctuations)

## Initial state “CGC”

VS

- Driven by initial momentum correlations
- Pre-exist before collisions or develops very soon after
- High-multiplicities are not required, but allowed
- Momentum correlations are present for “round” p

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# COLLECTIVITY: TUNING IN

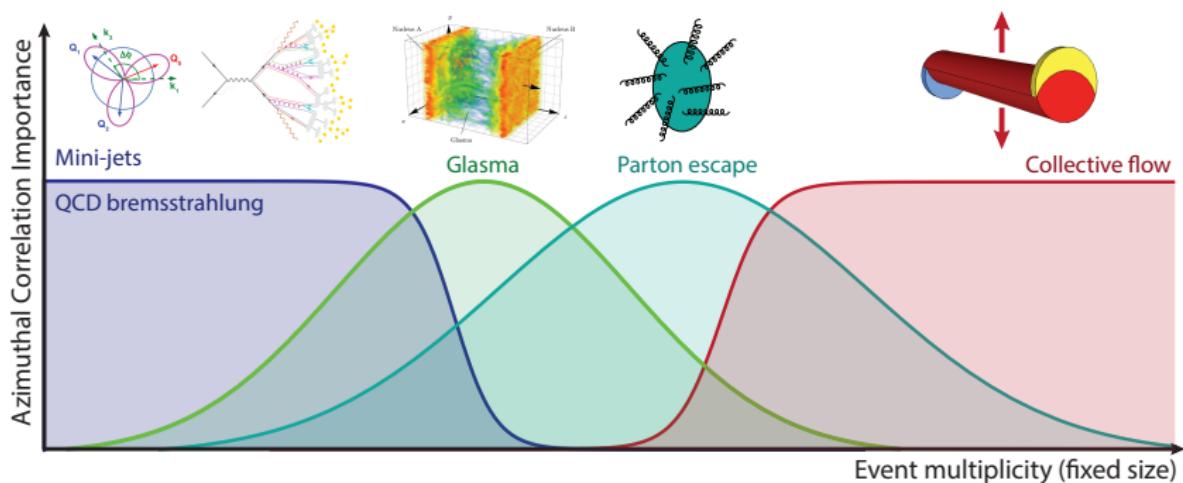
Initial state

Final state



control parameters

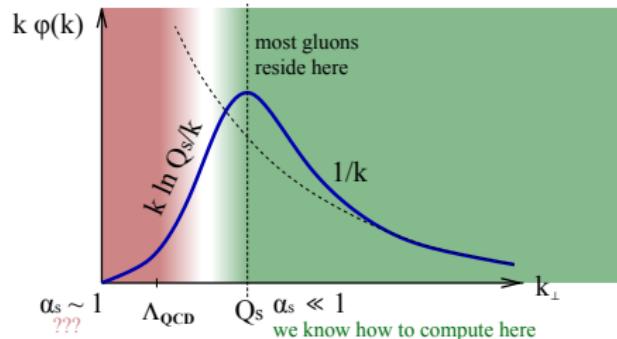
multiplicity, transverse momentum, system size



Adapted from M. Strickland's talk on QM '18

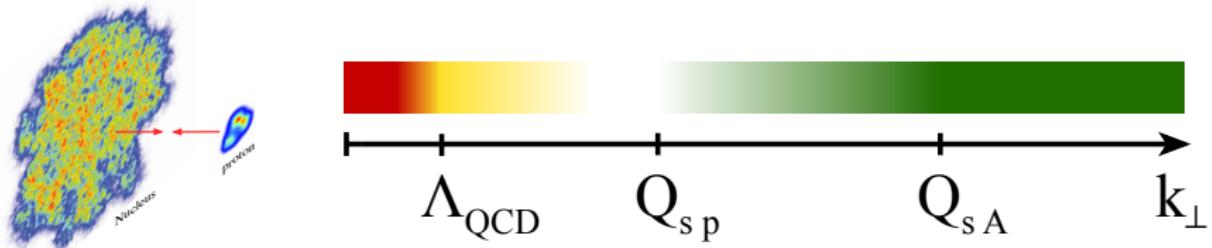
# SATURATION REGIME/CGC

- High energy  $\sim$  high gluon density  
 $\sim$  formation of semi-hard scale,  $Q_s$
- Particle production is dominated by  
 $k_\perp \sim Q_s$
- Weak coupling methods can be applied  
 $\alpha_s(Q_s) \ll 1$
- Still non-perturbative, as fields are strong,  $A \sim \frac{1}{g} \rightsquigarrow$  non-linearity is important
- Actual analytical calculations can be very hard



# WHAT DO WE KNOW ANALYTICALLY?

Asymmetric collisions, when  $Q_s$  of projectile  $\neq Q_s$  of target, is the easiest case.



Single inclusive production

- In general

$$\frac{dN}{d^3k} = \frac{1}{\alpha_s} f\left(\frac{Q_{sp}^2}{k_\perp^2}, \frac{Q_{sA}^2}{k_\perp^2}\right)$$

$f\left(\frac{Q_{sp}^2}{k_\perp^2}, \frac{Q_{sA}^2}{k_\perp^2}\right)$  is known only numerically; for large  $k_\perp \gg Q_{sA}^2$ :  $\frac{dN}{d^3k} = \frac{1}{\alpha_s} \frac{Q_{sp}^2}{k_\perp^2} \frac{Q_{sA}^2}{k_\perp^2} f^{(1,1)}$

A. Krasnitz, R. Venugopalan, arXiv:9809433

E. Kuraev, L. Lipatov, V. Fadin, 77

- If  $k_\perp > Q_{sp}$ ,

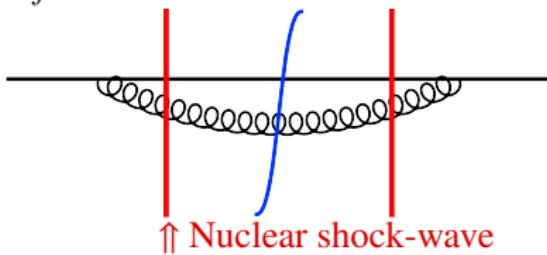
$$\frac{dN}{d^3k} = \frac{1}{\alpha_s} \frac{Q_{sp}^2}{k_\perp^2} f^{(1)}\left(\frac{Q_{sA}^2}{k_\perp^2}\right) + \frac{1}{\alpha_s} \left(\frac{Q_{sp}^2}{k_\perp^2}\right)^2 f^{(2)}\left(\frac{Q_{sA}^2}{k_\perp^2}\right) + \dots$$

Functions  $f^{(n)}$  are calculable!

# SINGLE INCLUSIVE PRODUCTION

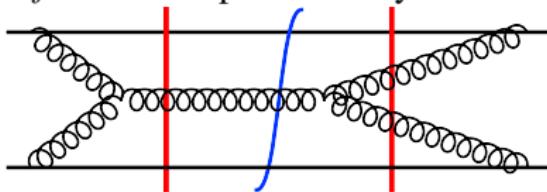
$$\frac{dN}{d^3k} = \frac{1}{\alpha_s} \frac{Q_{sp}^2}{k_\perp^2} f^{(1)} \left( \frac{Q_{sp}^2}{k_\perp^2} \right) + \frac{1}{\alpha_s} \left( \frac{Q_{sp}^2}{k_\perp^2} \right)^2 f^{(2)} \left( \frac{Q_{sp}^2}{k_\perp^2} \right) + \dots$$

- $f^{(1)}$  is known since '98



Y. V. Kovchegov and A. H. Mueller, arXiv:hep-ph/9802440  
A. Dumitru and L. D. McLerran, arXiv:hep-ph/0105268  
J.-P. Blaizot, F. Gelis, R. Venugopalan, arXiv:0402256

- $f^{(2)}$ : no complete result yet



I. Balitsky, arXiv:hep-ph/0409314  
G. A. Chirilli, Y. V. Kovchegov, and D. E. Wertepny, arXiv:1501.03106

# DOUBLE INCLUSIVE PRODUCTION

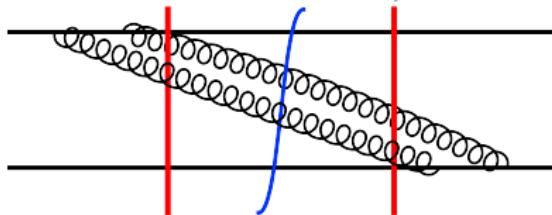
$$\frac{d^2N}{d^3k d^3p} = \frac{1}{\alpha_s^2} Q_{sp}^4 h^{(1)}(Q_{sA}) + \frac{1}{\alpha_s^2} Q_{sp}^6 h^{(2)}(Q_{sA}) + \dots$$

Momentum dependence is omitted to simplify notation

- Dilute-dilute “Glasma” graph:  $\frac{d^2N}{d^3k d^3p} = \frac{1}{\alpha_s^2} Q_{sp}^4 Q_{sA}^4 h^{(1,1)}$

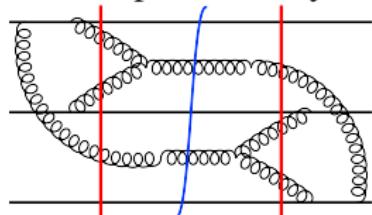
A. Dumitru, F. Gelis, L. McLerran and R. Venugopalan, arXiv:0804.3858

- $h^{(1)}$  is known since '12 ; invariant under  $(k_\perp \rightarrow -k_\perp)$



A. Kovner and M. Lublinsky, arXiv:1211.1928  
Y. V. Kovchegov and D. E. Wertheimer, arXiv:1212.1195

- $h^{(2)}$ : no complete result yet



L. McLerran and V. S., arXiv:1611.09870;  
Yu. Kovchegov and V. S., arXiv:1802.08166

# WHAT DOES PRESENCE OF ODD HARMONICS MEAN?

- Double inclusive production

$$\frac{d^2N}{d^2\mathbf{k}_1 dy_1 d^2\mathbf{k}_2 dy_2} = \frac{d^2N}{\mathbf{k}_1 d\mathbf{k}_1 dy_1 \mathbf{k}_2 d\mathbf{k}_2 dy_2} \\ \times \left( 1 + 2v_2^2\{2\} \cos 2(\phi_1 - \phi_2) + 2v_3^2\{2\} \cos 3(\phi_1 - \phi_2) + \dots \right)$$

- A non-vanishing  $v_3^2\{2\}$

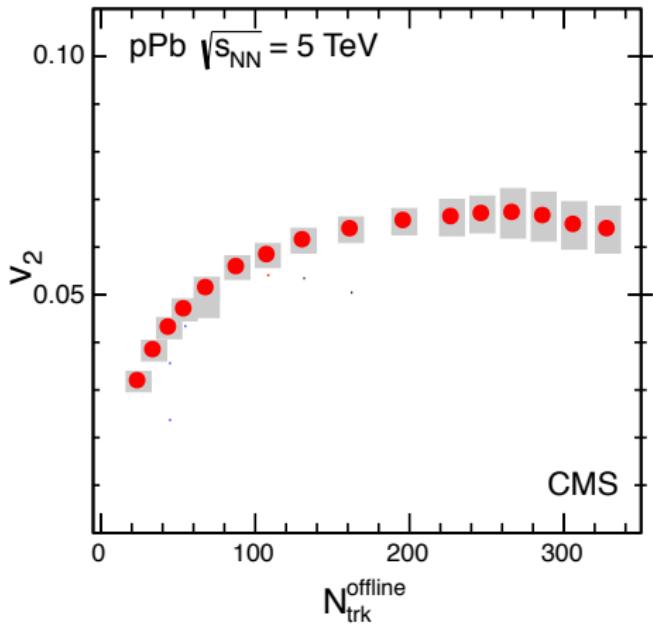
$$\int_0^{2\pi} d\Delta\phi \cos 3\Delta\phi \frac{d^2N}{d^2k_1 d^2k_2}(\delta\phi) = \int_0^\pi d\Delta\phi \cos 3\Delta\phi \frac{d^2N}{d^2k_1 d^2k_2}(\delta\phi) - \int_0^\pi d\Delta\phi \cos 3\Delta\phi \frac{d^2N}{d^2k_1 d^2k_2}(\delta\phi + \pi) \\ = \int_0^\pi d\Delta\phi \cos 3\Delta\phi \left[ \frac{d^2N}{d^2k_1 d^2k_2}(\underline{k}_1, \underline{k}_2) - \frac{d^2N}{d^2k_1 d^2k_2}(\underline{k}_1, -\underline{k}_2) \right]$$

- Therefore, non-zero  $v_3 \leadsto$

$$\frac{d^2N}{d^2k_1 d^2k_2}(\underline{k}_1, \underline{k}_2) \neq \frac{d^2N}{d^2k_1 d^2k_2}(\underline{k}_1, -\underline{k}_2)$$

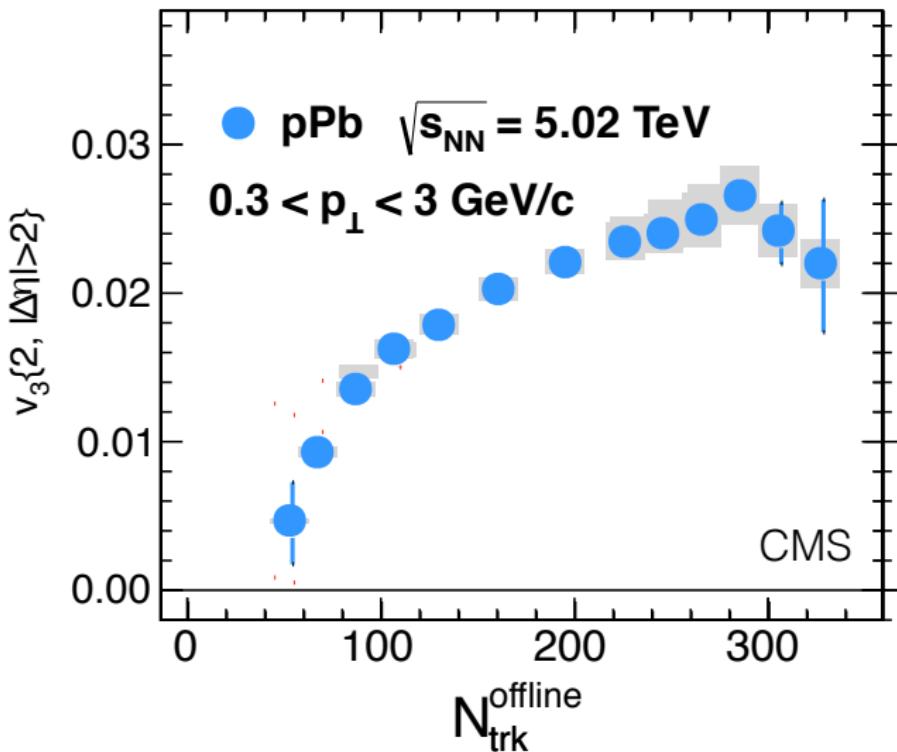
and is absent in “Glasma” graph and  $h^{(1)}$

# EXPERIMENTAL DATA: $v_2\{2\}$



$$\frac{d^2N}{d\mathbf{k}_1 dy_1 d\mathbf{k}_2 dy_2} = \frac{d^2N}{\mathbf{k}_1 d\mathbf{k}_1 dy_1 \mathbf{k}_2 d\mathbf{k}_2 dy_2} \left( 1 + 2v_2^2\{2\} \cos 2(\phi_1 - \phi_2) + 2v_3^2\{2\} \cos 3(\phi_1 - \phi_2) + \dots \right)$$

## EXPERIMENTAL DATA: $v_3\{2\}$



- Suppressed compared to  $v_2$ , but non-zero!

# A CONUNDRUM FOR SATURATION

Can saturation dynamics account  
for observed long-range rapidity correlations  
with non-zero odd azimuthal harmonics?

# INSPIRATION FROM SINGLE TRANSVERSE SPIN ASYMMETRY

- Consider single gluon production

$$\frac{d\sigma}{d^2k} \sim |M(\underline{k})|^2 = \int d^2x d^2y e^{-i\underline{k}\cdot(\underline{x}-\underline{y})} M(\underline{x}) M^*(\underline{y})$$

- Amplitude may have two contributions

$$M(\underline{x}) = \textcolor{blue}{M}_1(\underline{x}) + \textcolor{red}{M}_3(\underline{x}) + \dots$$

- Asymmetry under  $\underline{k} \rightarrow -\underline{k}$  would mean that

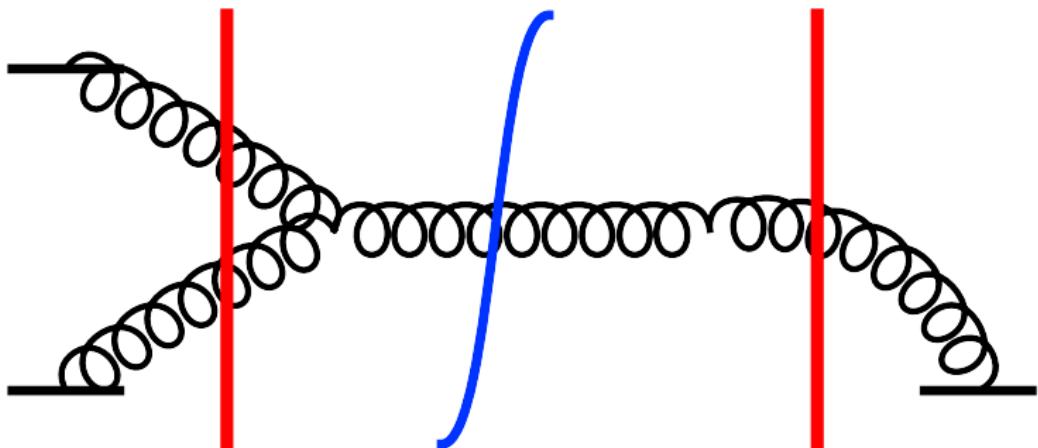
$$\textcolor{blue}{M}_1(\underline{x}) \textcolor{red}{M}_3^*(\underline{y}) + \textcolor{red}{M}_3(\underline{x}) \textcolor{blue}{M}_1^*(\underline{y}) = -\textcolor{blue}{M}_1(\underline{y}) \textcolor{red}{M}_3^*(\underline{x}) - \textcolor{red}{M}_3(\underline{y}) \textcolor{blue}{M}_1^*(\underline{x})$$

$\rightsquigarrow \textcolor{blue}{M}_1(\underline{x}) \textcolor{red}{M}_3^*(\underline{y})$  is imaginary

$\rightsquigarrow$  Phase difference between  $\textcolor{blue}{M}_1$  and  $\textcolor{red}{M}_3$  in coordinate space

*In coordinate space, but not dissimilar from STSA  
S. Brodsky, D. S. Hwang, Y. Kovchegov, I. Schmidt, M. Sievert, arXiv:1304.5237*

## NATURAL CANDIDATE

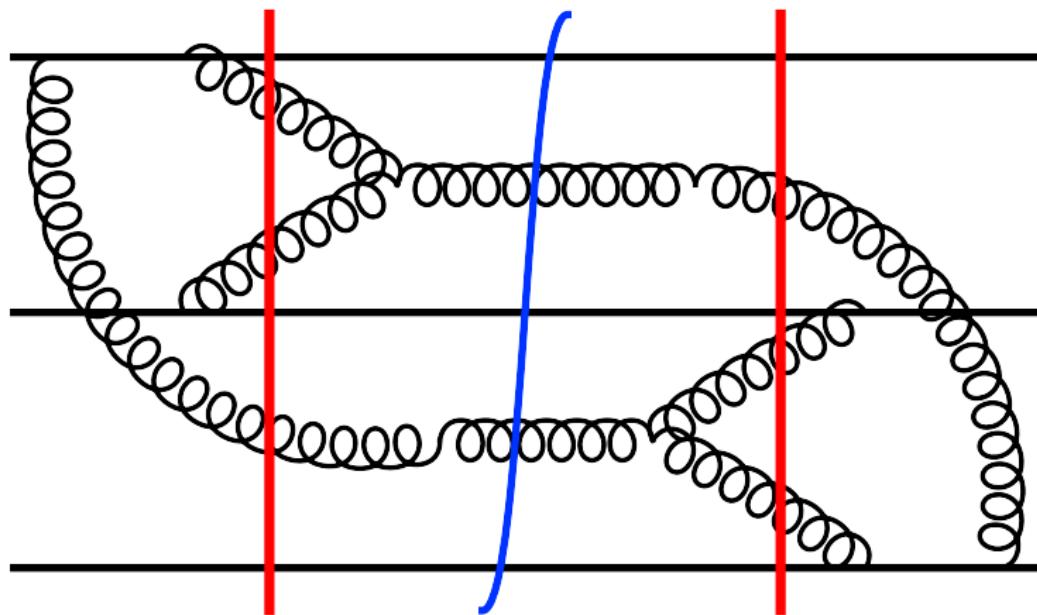


$M_3$

$M_1$

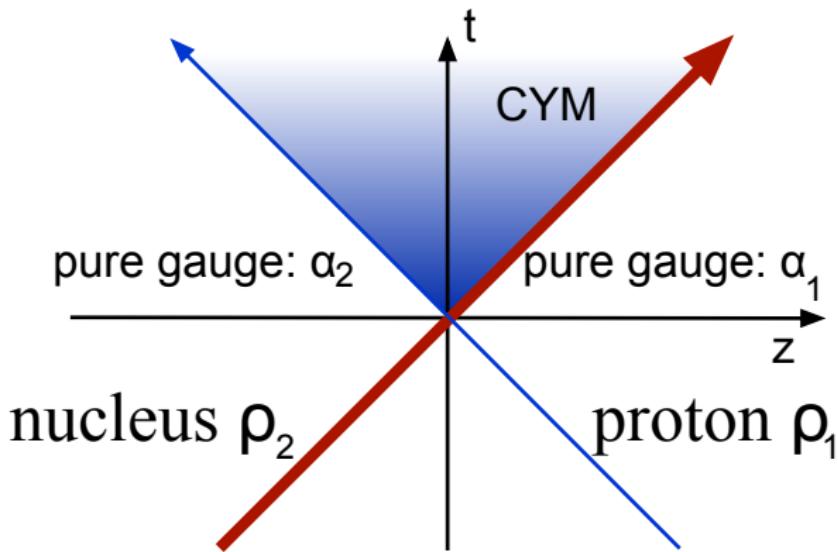
- Vanishes for single-inclusive production after performing average with respect to projectile configurations...

# DOUBLE INCLUSIVE GLUON PRODUCTION



- Non-zero!

# CLASSICAL YANG-MILLS



- Just after collision,  $\tau \rightarrow 0+$ , initial conditions are known (Fock-Schwinger gauge  $A_\tau = 0$ )  
*A. Kovner, L. McLerran, H. Weigert, arXiv:9506320*
- In forward light-cone  $[D_\mu, F^{\mu\nu}] = 0$
- Solve equations perturbatively in  $\rho_1$ ; use LSZ

# GLUON PRODUCTION

- Leading order and the first saturation correction

$$\frac{dN^{\text{even}}(\underline{k})}{d^2kdy} [\rho_p, \rho_t] = \frac{2}{(2\pi)^3} \frac{\delta_{ij}\delta_{lm} + \epsilon_{ij}\epsilon_{lm}}{k^2} \Omega_{ij}^a(\underline{k}) [\Omega_{lm}^a(\underline{k})]^*$$

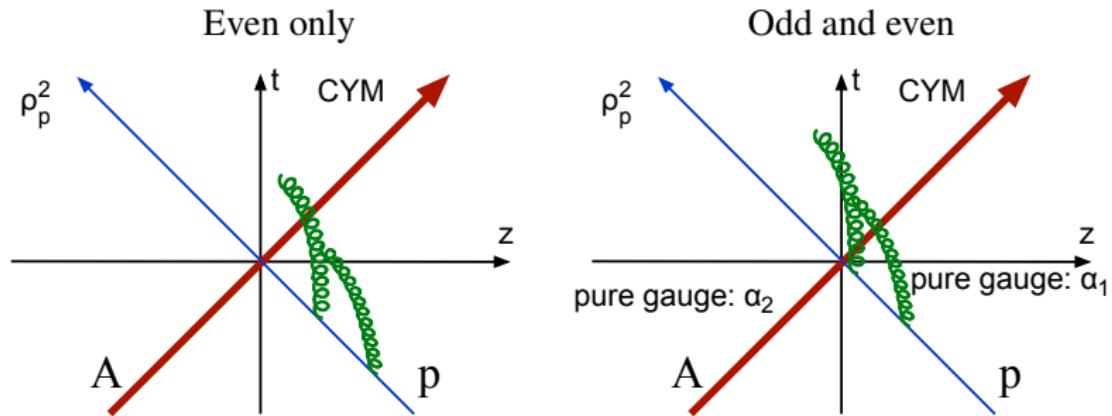
$$\begin{aligned} \frac{dN^{\text{odd}}(\underline{k})}{d^2kdy} [\rho_p, \rho_T] = & \frac{2}{(2\pi)^3} \text{Im} \left\{ \frac{g}{\underline{k}^2} \int \frac{d^2l}{(2\pi)^2} \frac{\text{Sign}(\underline{k} \times \underline{l})}{l^2 |\underline{k} - \underline{l}|^2} f^{abc} \Omega_{ij}^a(\underline{l}) \Omega_{mn}^b(\underline{k} - \underline{l}) [\Omega_{rp}^c(\underline{k})]^* \times \right. \\ & \left. \left[ (\underline{k}^2 \epsilon^{ij} \epsilon^{mn} - \underline{l} \cdot (\underline{k} - \underline{l})(\epsilon^{ij} \epsilon^{mn} + \delta^{ij} \delta^{mn})) \epsilon^{rp} + 2\underline{k} \cdot (\underline{k} - \underline{l}) \epsilon^{ij} \delta^{mn} \delta^{rp} \right] \right\} \end{aligned}$$

Here  $\delta_{ij}\Omega_{ij} = \Omega_{xx} + \Omega_{yy}$  and  $\epsilon_{ij}\Omega_{ij} = \Omega_{xy} - \Omega_{yx}$  and

$$\Omega_{ij}^a(\mathbf{x}_\perp) = g \underbrace{\left[ \frac{\partial_i}{\partial^2} \overbrace{\rho^b(\mathbf{x}_\perp)}^{\text{val. sour.}} \right] \partial_j}_{\text{valence sources rotated by the target}} \overbrace{U^{ab}(\mathbf{x}_\perp)}^{\text{target W line}}$$

$\frac{dN^{\text{odd}}(\underline{k})}{d^2kdy} [\rho_p, \rho_T]$  is suppressed by extra  $\alpha_s \rho_p$

# BEYOND CLASSICAL APPROXIMATION: A SHORT DETOUR



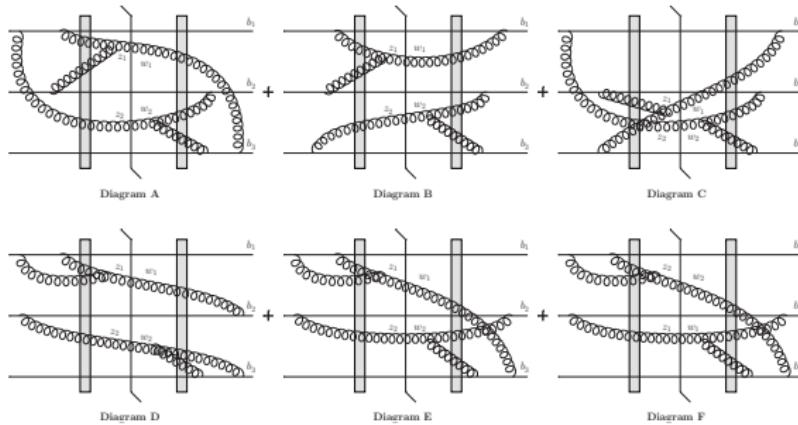
- In this particular gauge:  
in classical approximation,  $v_3$  requires some degree of final state interaction
- Do we have odd azimuthal component of two parton correlation function  
in hadron wave function?!
- There is hint that the answer is yes,  
more work has to be done...

*A. Kovner, M. Lublinsky, & V. Skokov, arXiv:1612.07790*

# ALTERNATIVE APPROACH

- This was obtained in Fock-Schwinger gauge  $A_\tau = 0$ ;  
the gauge is singular; defined in coordinate space.
- Motivation to compute in global gauge  $A^+ = 0$

*Yu. Kovchegov and V. S., arXiv:1802.08166*

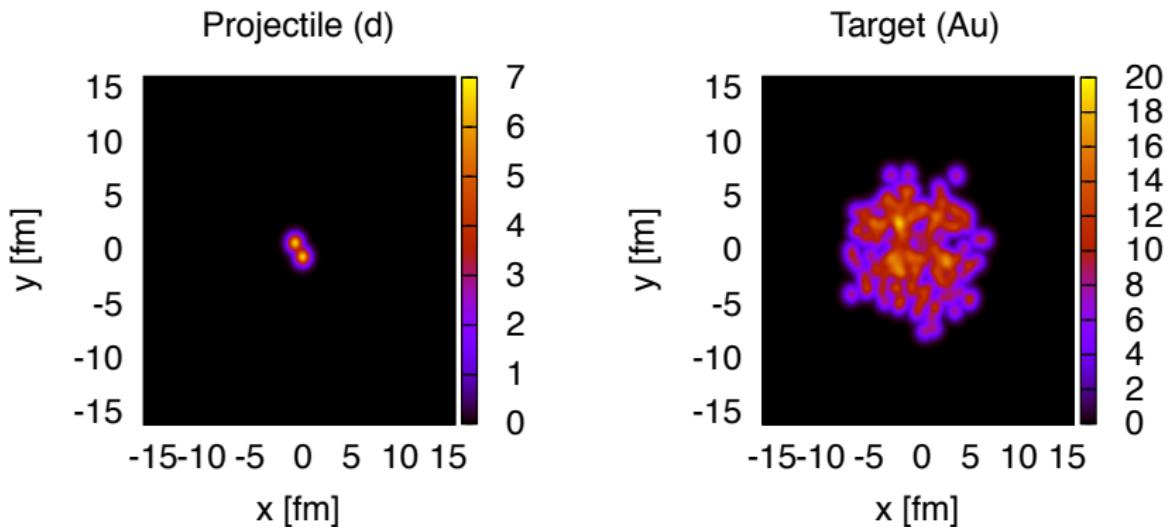


- Reproduces result obtained in Fock-Schwinger gauge!
- In Golec-Biernat-Wusthoff model & Large  $N_c$  & at high momentum:

$$\begin{aligned}
 \frac{d\sigma_{odd}}{d^2k_1 dy_1 d^2k_2 dy_2} = & \frac{1}{[2(2\pi)^3]^2} \int d^2B d^2b \left[ T_1(\underline{B} - \underline{b}) \right]^3 g^8 Q_{s0}^6(b) \frac{1}{\underline{k}_1^6 \underline{k}_2^6} \\
 & \times \left\{ \underbrace{\left[ \frac{(\underline{k}_1^2 + \underline{k}_2^2 + \underline{k}_1 \cdot \underline{k}_2)^2}{(\underline{k}_1 + \underline{k}_2)^6} - \frac{(\underline{k}_1^2 + \underline{k}_2^2 - \underline{k}_1 \cdot \underline{k}_2)^2}{(\underline{k}_1 - \underline{k}_2)^6} \right]}_A + \underbrace{\frac{10 c^2}{(2\pi)^2} \frac{1}{\Lambda^2} \frac{\underline{k}_1 \cdot \underline{k}_2}{\underline{k}_1 \underline{k}_2}}_B \right. \\
 & \left. + \underbrace{\frac{1}{4\pi} \frac{\underline{k}_1^4}{\Lambda^4} [\delta^2(\underline{k}_1 - \underline{k}_2) - \delta^2(\underline{k}_1 + \underline{k}_2)]}_C \right\}
 \end{aligned}$$

*Yu. Kovchegov and V. S., arXiv:1802.08166*

# PHENOMENOLOGICAL APPLICATION



- Sample nucleon positions+IP-Sat for color density distribution

Kowalski, Teaney, '03  
Schenke, Tribedy, Venugopalan, '12

- Negative binomial distribution from first principles – not an input!

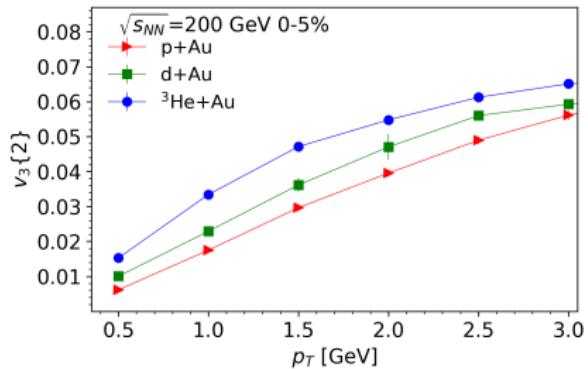
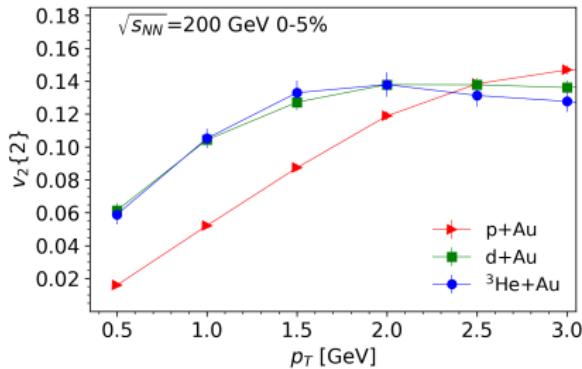
F. Gelis, T. Lappi, L. McLerran arXiv:0905.3234  
A. Kovner & V. S. arXiv:1805.09296

- In good agreement with STAR d+Au multiplicity distribution

M. Mace, V. S., P. Tribedy, & R. Venugopalan, arXiv:1805.09342, in production in Phys. Rev. Lett.

# NUMERICAL RESULTS FOR p-AU, d-AU AND $^3\text{He}$ -AU AT RHIC

- Hierarchy of anisotropies across systems

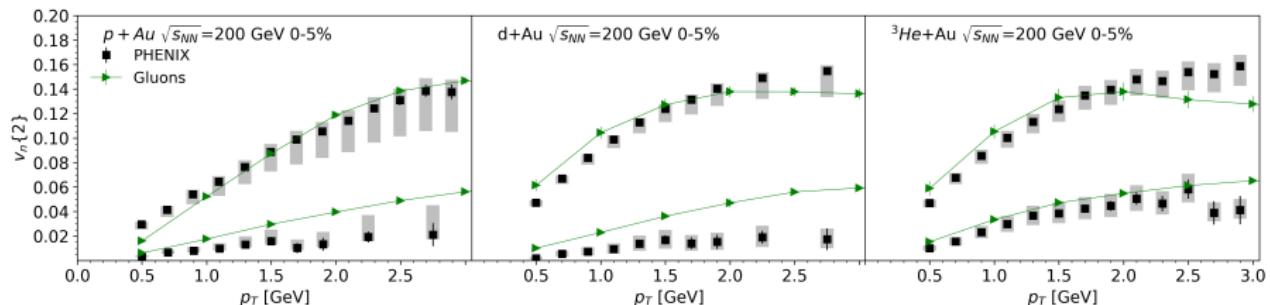


- System size dependence observed at STAR captured in the framework  
no need for hydrodynamic response to initial geometry!

M. Mace, V. S., P. Tribedy, & R. Venugopalan, arXiv:1805.09342, in production in Phys. Rev. Lett.

# NUMERICAL RESULTS FOR p-AU, d-AU AND $^3\text{He}$ -AU AT RHIC

- Hierarchy of anisotropies across systems



- Goes beyond qualitative description

*M. Mace, V. S., P. Tribedy, & R. Venugopalan, arXiv:1805.09342, in production in Phys. Rev. Lett.*

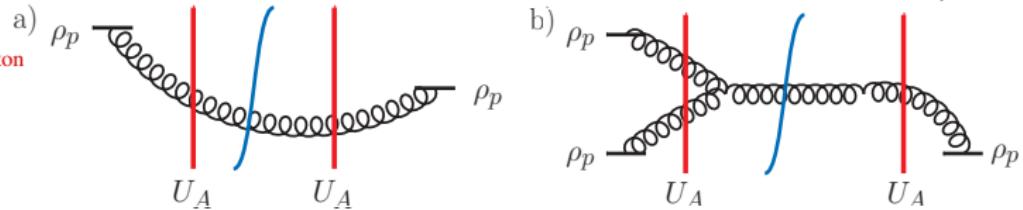
# CGC PERSPECTIVE ON $v_3$

- Leading order and the first saturation correction

$$a) \frac{dN^{\text{even}}(\underline{k})}{d^2k dy} [\rho_p, \rho_t] = \frac{2}{(2\pi)^3} \frac{\delta_{ij}\delta_{lm} + \epsilon_{ij}\epsilon_{lm}}{k^2} \Omega_{ij}^a(\underline{k}) [\Omega_{lm}^a(\underline{k})]^*$$

$$b) \frac{dN^{\text{odd}}(\underline{k})}{d^2k dy} [\rho_p, \rho_T] = \frac{2}{(2\pi)^3} \text{Im} \left\{ \frac{g}{k^2} \int \frac{d^2l}{(2\pi)^2} \frac{\text{Sign}(\underline{k} \times \underline{l})}{l^2 |\underline{k} - \underline{l}|^2} f^{abc} \Omega_{ij}^a(\underline{l}) \Omega_{mn}^b(\underline{k} - \underline{l}) [\Omega_{rp}^c(\underline{k})]^* \right. \\ \left. \left[ (\underline{k}^2 \epsilon^{ij} \epsilon^{mn} - \underline{l} \cdot (\underline{k} - \underline{l}) (\epsilon^{ij} \epsilon^{mn} + \delta^{ij} \delta^{mn})) \epsilon^{rp} + 2\underline{k} \cdot (\underline{k} - \underline{l}) \epsilon^{ij} \delta^{mn} \delta^{rp} \right] \right\}$$

Recall that  $\Omega \propto \rho_{\text{proton}}$



- Odd azimuthal harmonics is a sign of emerging coherence in proton wave function:  
the first saturation correction!

**Non-zero long-range odd harmonics in high energy p-A is evidence of saturation!**

# MULTIPLICITY DEPENDENCE: SCALING ARGUMENT

- Physical two-particle anisotropy coefficients can be simply expressed as

$$v_n^2\{2\}(N_{\text{ch}}) = \int \mathcal{D}\rho_p \mathcal{D}\rho_t W[\rho_p] W[\rho_t] |\mathcal{Q}_n[\rho_p, \rho_t]|^2 \delta\left(\frac{dN}{dy}[\rho_p, \rho_t] - N_{\text{ch}}\right)$$

with

$$\mathcal{Q}_{2n}[\rho_p, \rho_t] = \frac{\int_{p_1}^{p_2} k_\perp dk_\perp \frac{d\phi}{2\pi} e^{i2n\phi} \frac{dN^{\text{even}}(k)}{d^2kdy} [\rho_p, \rho_t]}{\int_{p_1}^{p_2} k_\perp dk_\perp \frac{d\phi}{2\pi} \frac{dN^{\text{even}}(k)}{d^2kdy} [\rho_p, \rho_t]}, \quad \mathcal{Q}_{2n+1}[\rho_p, \rho_t] = \frac{\int_{p_1}^{p_2} k_\perp dk_\perp \frac{d\phi}{2\pi} e^{i(2n+1)\phi} \frac{dN^{\text{odd}}(k)}{d^2kdy} [\rho_p, \rho_t]}{\int_{p_1}^{p_2} k_\perp dk_\perp \frac{d\phi}{2\pi} \frac{dN^{\text{odd}}(k)}{d^2kdy} [\rho_p, \rho_t]}$$

- High multiplicity is driven by fluctuations in  $\rho_p$
- To study multiplicity dependence, rescale  $\rho_p \rightarrow c \rho_p$
- Under this rescaling:

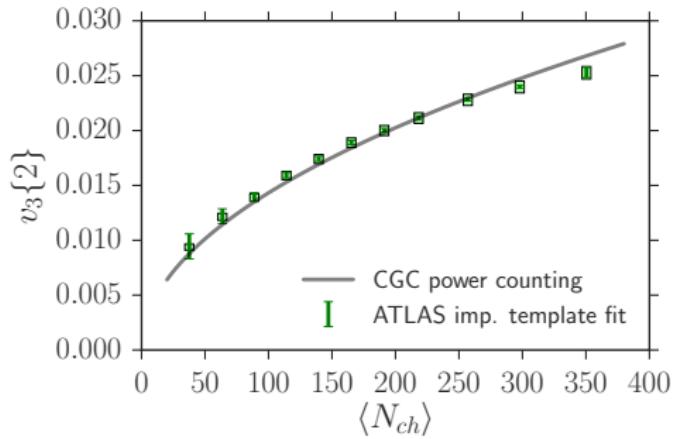
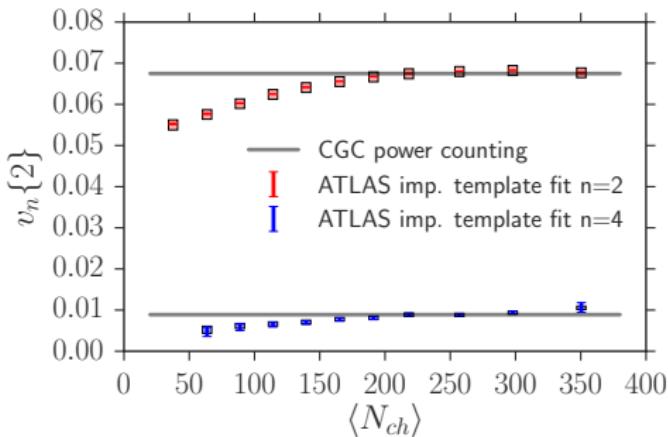
$$\frac{dN}{dy} \rightarrow c \frac{dN}{dy}; \quad v_{2n}^2\{2\} \rightarrow v_{2n}^2\{2\}; \quad v_{2n+1}^2\{2\} \rightarrow c v_{2n+1}^2\{2\}$$

- Therefore in the first approximation:  $v_{2n}\{2\}$  is independent of multiplicity

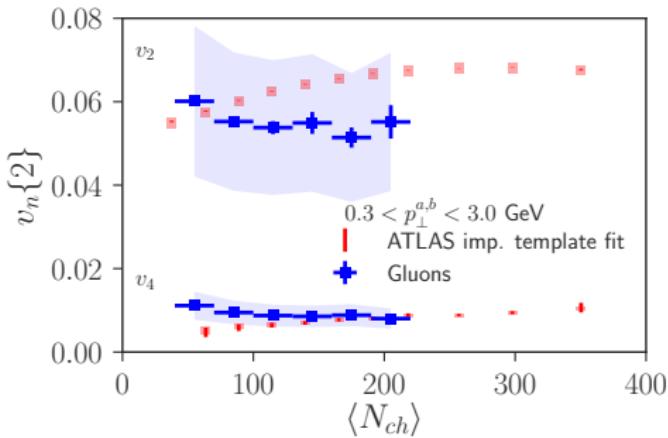
$$v_{2n+1}\{2\} \propto \sqrt{\frac{dN}{dy}}$$

# MULTIPLICITY DEPENDENCE: SCALING ARGUMENT

M. Mace , V. S., P. Tribedy, & R. Venugopalan, arXiv:1807.00825



# MULTIPLICITY DEPENDENCE: NUMERICAL RESULT



M. Mace , V. S., P. Tribedy, & R. Venugopalan, arXiv:1807.00825



- Multiplicity dependence of integrated  $v_3$   
is beyond our computational resources

# CONCLUSIONS

- Odd azimuthal harmonics
  - are an inherent property of particle production in the saturation framework
- Non-zero long range in  $y$  odd azimuthal harmonics  $\Leftrightarrow$  evidence of saturation
- Phenomenological applications:
  - able to describe system size hierarchy of  $v_2$  and  $v_3$  in p-Au, d-Au,  $^3\text{He}$ -Au
  - able to describe multiplicity dependence in p-A at LHC
- Check on systematic uncertainties is required

*Dilute-dense approximation: high density effects need to be quantified  
Fragmentation*

...



# CUMULANTS & PHENOMENOLOGICAL CONCLUSION I

- Average number of gluons

$$\kappa_1 = \frac{N_c^2 - 1}{8\pi} \underbrace{S_\perp \mu_p^2}_{\mathfrak{D}} \ln \frac{k_{\min}^2}{\Lambda^2}$$

- Higher order cumulants

$$\kappa_{n \geq 2} = \left. \frac{\partial}{\partial t^n} \ln G_{\text{LO}}(t) \right|_{t=0} = (n-2)! \frac{(N_c^2 - 1) S_\perp \Lambda^2}{8\pi} \left( \frac{\mu_p^2 \mathfrak{D}}{\Lambda^2} \right)^n$$

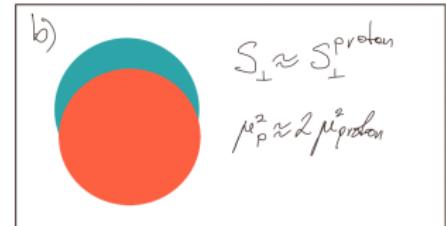
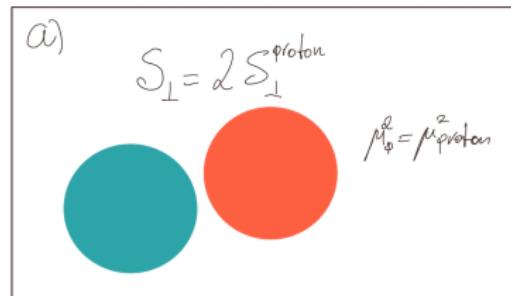
- Properties ( $\Lambda^2 = S_\perp^{\text{proton}}$ )

- $\kappa_1$  is a function of  $S_\perp \mu_p^2$
- Consider configurations a) and b):  
 $\kappa_1[a] = \kappa_1[b]$

$$\kappa_n[b] \propto 2 S_\perp^{\text{proton}} (\mu_p^{\text{proton}})^{2n}$$

$$\kappa_n[b] \propto 2^n S_\perp^{\text{proton}} (\mu_p^{\text{proton}})^{2n}$$

High multiplicity tail  $\equiv$   
 $\equiv$  configurations with  
overlapping nucleons



## MORE FUNCTIONALS

- BE is property of projectile
- Need for effective theory of gluons in projectile
- Constraint effective action for projectile gluon distribution

$$e^{-V_{\text{eff}}[\eta(\underline{q})]} = \frac{1}{Z_p} \int \mathcal{D}\rho_p \underbrace{W(\rho_p)}_{\text{all possible fluct.}} \underbrace{\delta\left(\eta(\underline{q}) - \frac{g^2 \text{tr}|A^+(\underline{q})|^2}{\langle g^2 \text{tr}|A^+(\underline{q})|^2 \rangle}\right)}_{\text{keeping only interesting stuff}}$$

$$A^+(\underline{q}) = g/q^2 \rho_p(q), \quad \langle g^2 \text{tr}|A^+(\underline{q})|^2 \rangle = \frac{1}{2}(N_c^2 - 1)S_\perp \frac{g^4 \mu_p^2}{q^4}$$

- Exact expression for effective potential (modulo  $S_\perp^{-1}$  corrections)

$$V_{\text{eff}}[\eta(\underline{q})] = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2 q}{(2\pi)^2} \left\{ \eta(\underline{q}) - 1 - \ln \eta(\underline{q}) \right\} \approx \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2 q}{(2\pi)^2} \frac{1}{2} \ln^2 \eta(\underline{q})$$

# LIOUVILLE POTENTIAL & HIGH MULTIPLICITY TAIL

- Back to generating function

$$G_{\text{LO}}(t) = \left\langle \exp \left[ t \int_{\Lambda}^{k_{\min}} \frac{d^2 q}{(2\pi)^2} \rho^a(-\underline{q}) \frac{\mathfrak{D}}{2q^2} \rho^a(\underline{q}) \right] \right\rangle_p$$

- In terms of effective potential

$$G_{\text{LO}}(t) = \int \mathcal{D}\eta \exp \left( -V_{\text{eff}}[\eta(q)] + \underbrace{\frac{1}{2}(N_c^2 - 1)S_{\perp} \int_{\Lambda}^{k_{\min}} \frac{d^2 q}{(2\pi)^2} t \frac{\mu_p^2 \mathfrak{D}}{q^2} \eta(\underline{q})}_{\text{reweighting! derivatives in } t \text{ probe Liouville potential}} \right)$$

- For large  $S_{\perp}$ : saddle point approximation

$$\eta_s(\underline{q}) = \begin{cases} \left(1 - t \frac{\mu_p^2 \mathfrak{D}}{q^2}\right)^{-1}, & \text{if } \Lambda \leq q \leq k_{\min} \\ 1, & \text{otherwise} \end{cases}$$

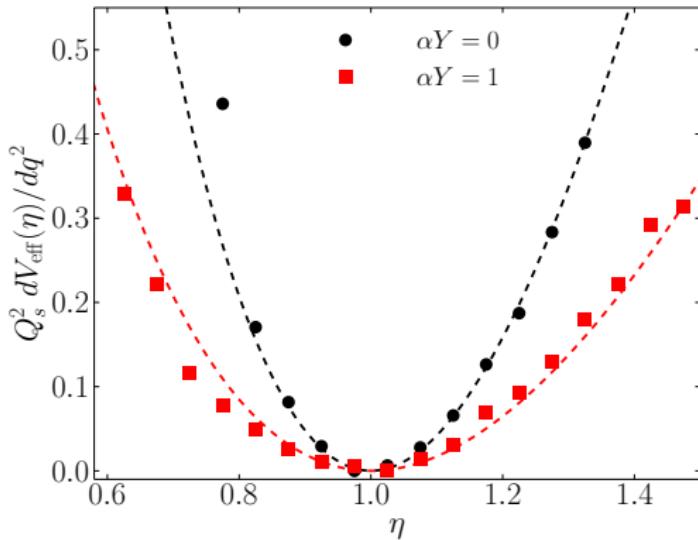
to yield

$$\ln G_{\text{LO}}(t) = \frac{1}{2}(N_c^2 - 1)S_{\perp} \int \frac{d^2 q}{(2\pi)^2} \ln \eta_s(\underline{q}) = -\frac{1}{2}(N_c^2 - 1)S_{\perp} \int_{\Lambda}^{k_{\min}} \frac{d^2 q}{(2\pi)^2} \ln \left(1 - t \frac{\mu_p^2 \mathfrak{D}}{q^2}\right)$$

- We recovered previously derived result. Origin of  $\ln \equiv$  Liouville's  $\ln!$

# LIOUVILLE POTENTIAL AND SMALL X EVOLUTION

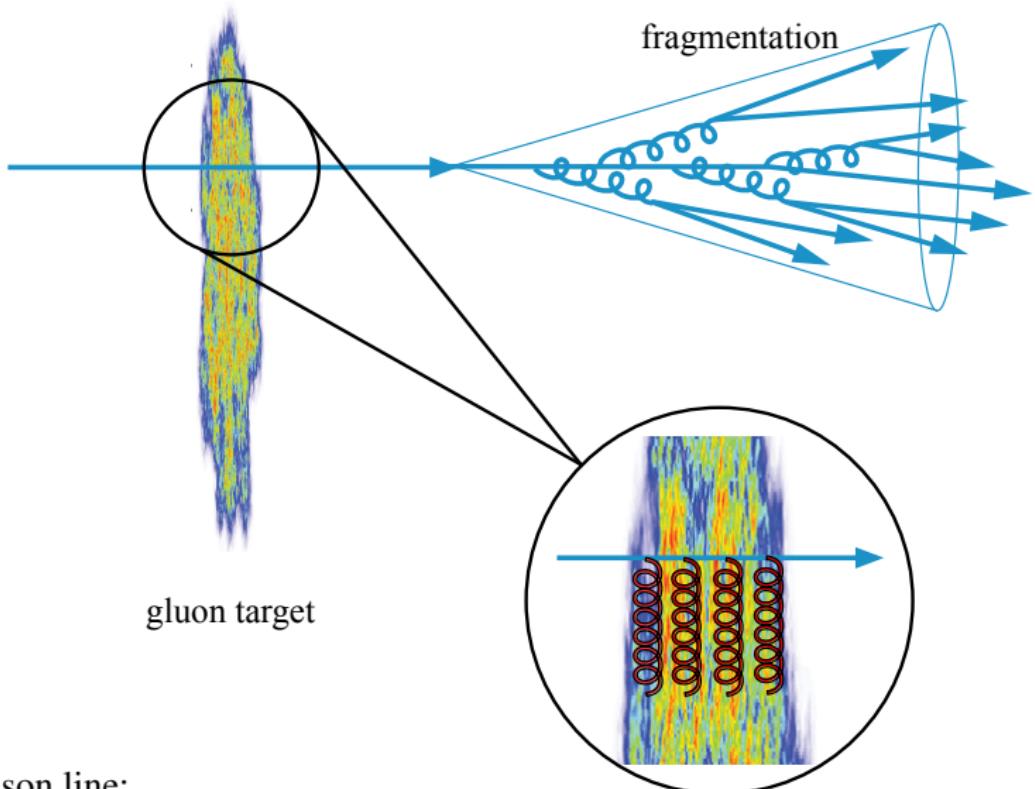
A. Dumitru & V. S.,  
arXiv:1704.05917



- Form does not change  $V_{\text{eff}}[\eta(\underline{q})] \approx \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2 q}{(2\pi)^2} \frac{1}{2} \ln^2 \eta(\underline{q})$
- $S_\perp \rightarrow S_\perp^{\text{eff}} \equiv \frac{S_\perp}{\sigma^2}$ :  
partially responsible for phenomenological parameter  $\sigma$
- C.f.  $P[\rho] \propto \exp \left[ -\frac{\rho^2}{2\sigma^2} \right]$  with  $\rho \equiv \ln Q_s^2 / \bar{Q}_s^2$

L. McLerran & P. Tribedy, arXiv:1508.03292

# MULTIPLE RESCATTERING



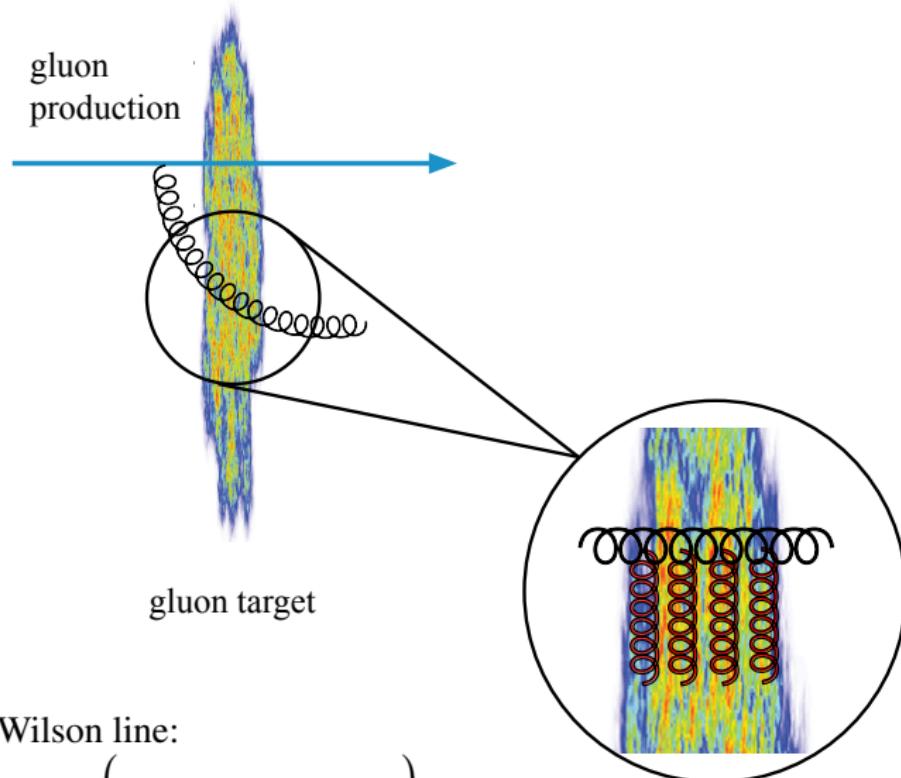
Fundamental Wilson line:

$$V(\mathbf{x}_\perp) = \mathcal{P} \exp \left( ig \int dx^+ A^-(x^+, \mathbf{x}_\perp) \right)$$

$\Leftarrow$

multiple rescattering

# GLUON PRODUCTION



Adjoint Wilson line:

$$U(\mathbf{x}_\perp) = \mathcal{P} \exp \left( ig \int dx^+ A^-_{\text{adj.}}(x^+, \mathbf{x}_\perp) \right) \quad \Leftarrow \quad \text{multiple rescattering}$$

## A POSSIBLE RESOLUTION

Odd contribution is buried somewhere in multiple  
rescattering i.e. in high order  $h^{(N \gg 1)}$



$$\frac{d^2N}{d^3k d^3p} = \frac{1}{\alpha_s^2} Q_{sp}^4 h^{(1)}(Q_{sA}) + \frac{1}{\alpha_s^2} Q_{sp}^6 h^{(2)}(Q_{sA}) + \dots$$

- Theoretically this is unsatisfactory
  - Phenomenologically this is problematic
    - $v_3\{2\}$  is observed in p-A
    - $v_3\{2\}$  is not much smaller than  $v_2\{2\}$
- ~  $v_3\{2\}$  must originate from rather low order corrections  
to the leading order dilute-dense production