Exploring the pion structure with a Minkowski space dynamical model

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Collaborators

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Motivation

Physical space-time = Minkowski space

Develop methods in continuous nonperturbative QCD within a given dynamical simple framework

Solve the Bethe-Salpeter bound state equation

Observables: spectrum, SL/TL momentum region

Relation BSA to LF Fock-space expansion of the hadron wf

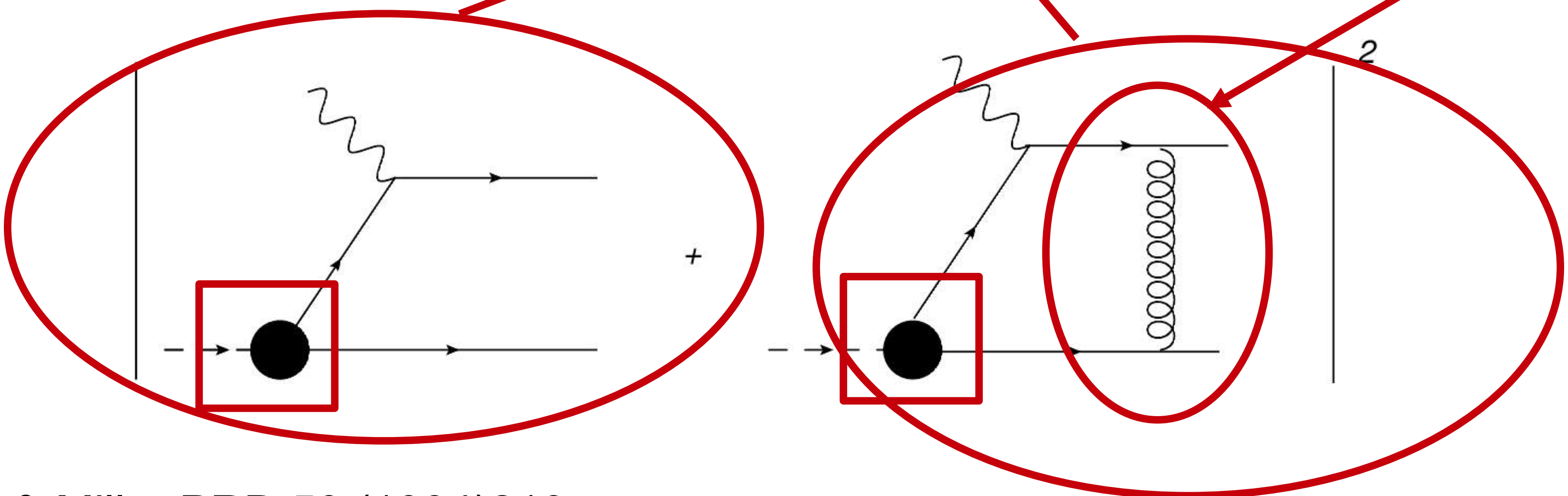
Problems to be addressed

Observables associated with the hadron structure in Minkowski Space obtainable from BSA

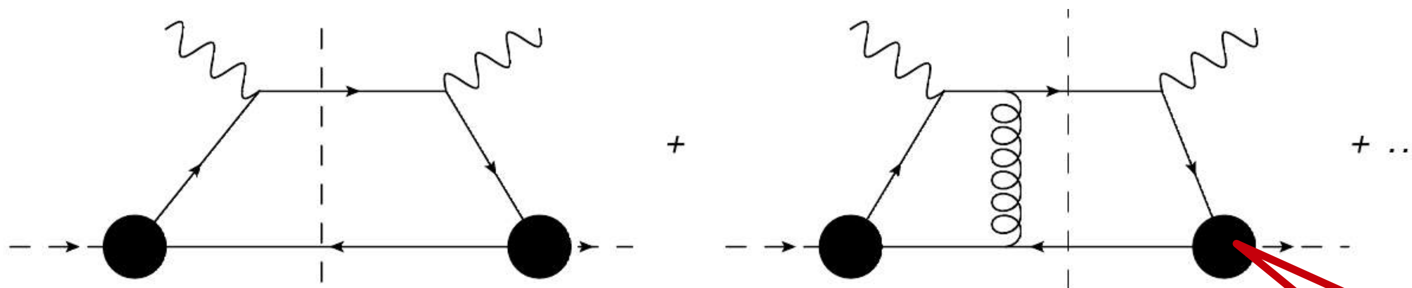
- parton distributions (pdfs)
- generalized parton distributions
- transverse momentum distributions (TMDs)
- Fragmentation functions
- SL and TL form factors ….
- Inversion Problem: Euclidean→Minkowski

TMDs & PDFs

FSI gluon exchange: T-odd



TF & Miller PRD 50 (1994)210



$$q^2 = q^+q^- - q^2_T$$

$$q^+ = q^0+q^3 \quad q^- = q^0-q^3$$

$q^- \longrightarrow$ infty
DIS

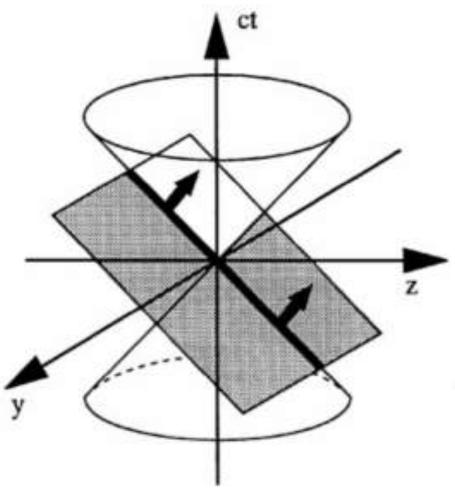
Bethe-Salpeter Amplitude @ $x^+=0$

Light-Front WF (LFWF)

basic ingredient in PDFs, GPDs and TMDs

$$\tilde{\Phi}(x,p) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\Phi(k,p)$$

$$p^\mu = p_1^\mu + p_2^\mu \qquad k^\mu = \frac{p_1^\mu - p_2^\mu}{2}$$



$$\tilde{\Phi}(x,p) = \langle 0|T\{\varphi_H(x^\mu/2)\varphi_H(-x^\mu/2)\}|p\rangle$$
$$= \theta(x^+)\langle 0|\varphi(\tilde{x}/2)e^{-iP^- x^+/2}\varphi(-\tilde{x}/2)|p\rangle e^{ip^- x^+/4} + \cdots$$
$$= \theta(x^+)\sum_{n,n'} e^{ip^- x^+/4}\langle 0|\varphi(\tilde{x}/2)|n'\rangle\langle n'|e^{-iP^- x^+/2}|n\rangle\langle n|\varphi(-\tilde{x}/2)|p\rangle + \cdots$$

$x^+ = 0$ only valence state remains! How to rebuilt the full BS amplitude?

Iterated Resolvents: Brodsky, Pauli, and Pinsky, Phys. Rep. 301, 299 (1998)

- **From the valence→ full Fock Space w-f:** Sales, et al. PRC61, 044003 (2000)

Main Tool: Nakanishi Integral Representation (NIR)

"Parametric representation for any Feynman diagram for interacting bosons, with a denominator carrying the overall analytical behavior in Minkowski space" (Nakanishi 1962)

Bethe-Salpeter amplitude

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_0^{\infty} d\gamma' \frac{g(\gamma', z')}{(\gamma' + \kappa^2 - k^2 - p.kz' - i\epsilon)^3}$$

$$\kappa^2 = m^2 - \frac{M^2}{4}$$

BSE in Minkowski space with NIR for bosons

Kusaka and Williams, PRD 51 (1995) 7026;

Light-front projection: integration in k⁻

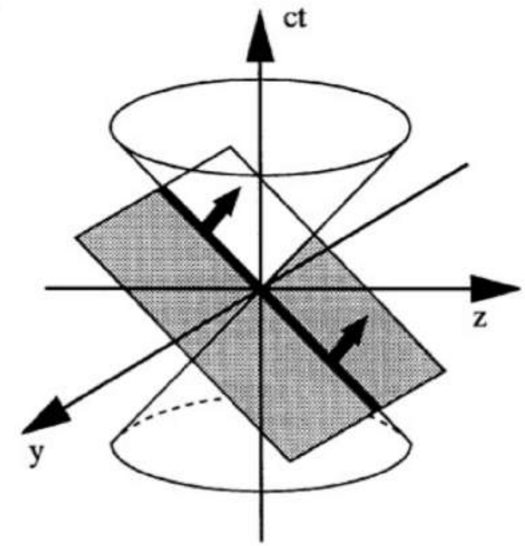
Carbonell&Karmanov EPJA27(2006)1;EPJA27(2006)11;

TF, Salme, Viviani PRD89(2014) 016010,…

Equivalent to Generalized Stietjes transform

Carbonell, TF, Karmanov PLB769 (2017) 418

LF wave function



$$\psi_{LF}(\gamma, z) = \frac{1}{4}(1 - z^2)\int_0^\infty \frac{g(\gamma', z)d\gamma'}{\Big[\gamma' + \gamma + z^2 m^2 + \kappa^2(1 - z^2)\Big]^2}$$

$$\gamma = k_\perp^2 \qquad\qquad z = 2x - 1$$

Generalized Stietjes transform and the LF valence wave function

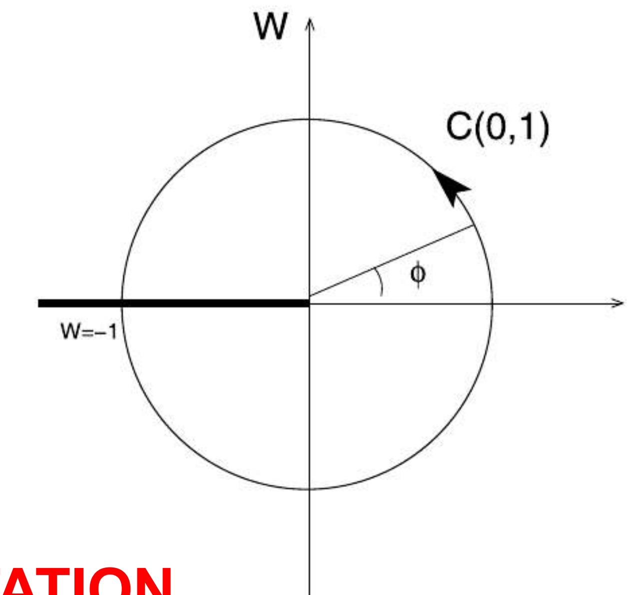
Jaume Carbonell, TF, Vladimir Karmanov PLB769 (2017) 418

$$\psi_{LF}(\gamma,z) = \frac{1-z^2}{4}\int_0^\infty \frac{g(\gamma',z)d\gamma'}{\left[\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2\right]^2}.$$

$$f(\gamma) \equiv \int_0^\infty d\gamma'\, L(\gamma,\gamma')g(\gamma') = \int_0^\infty d\gamma' \,\frac{g(\gamma')}{(\gamma'+\gamma+b)^2}$$

denoted symbolically as $f = \hat{L}\, g$.

$$g(\gamma) = \hat{L}^{-1} f = \frac{\gamma}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{i\phi}\, f(\gamma\, e^{i\phi} - b).$$



J.H. Schwarz, J. Math. Phys. 46 (2005) 014501,

UNIQUENESS OF THE NAKANISHI REPRESENTATION

PHENOMENOLOGICAL APPLICATIONS from the valence wf → BSA!

Two-Boson System: ground-state

Building a solvable model…

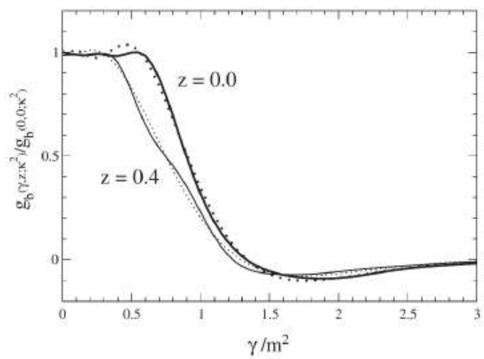
Nakanishi weight function

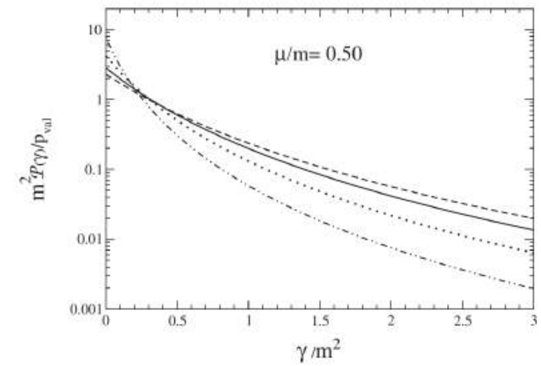
3+1 n=1

LADDER KERNEL

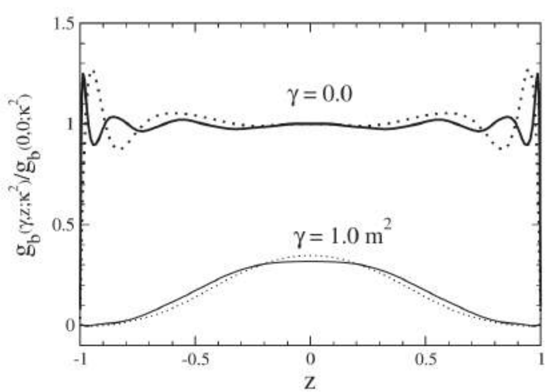
Valence wave function

3+1 n=1





$$\mu = 0.5 \ \ B/M = 1$$



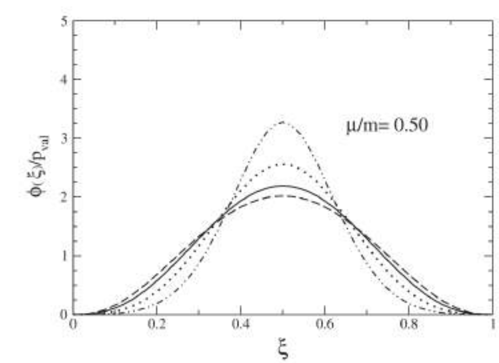


FIG. 3. The longitudinal LF distribution $\phi(\xi)$ for the valence component Eq. (34) vs the longitudinal-momentum fraction ξ for $\mu/m = 0.05,\ 0.15,\ 0.50$. Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. Recall that $\int_0^1 d\xi\phi(\xi) = P_{\rm val}$ (cf. Table III).

Karmanov, Carbonell, EPJA 27, 1 (2006)
Frederico, Salmè, Viviani PRD89, 016010 (2014)

(II) Valence LF wave function in impact parameter space

$$F(\xi, b)|_{b\to\infty} \to \mathrm{e}^{-b\sqrt{\kappa^2+(\xi-1/2)^2M^2}}\, f(\xi, b)$$

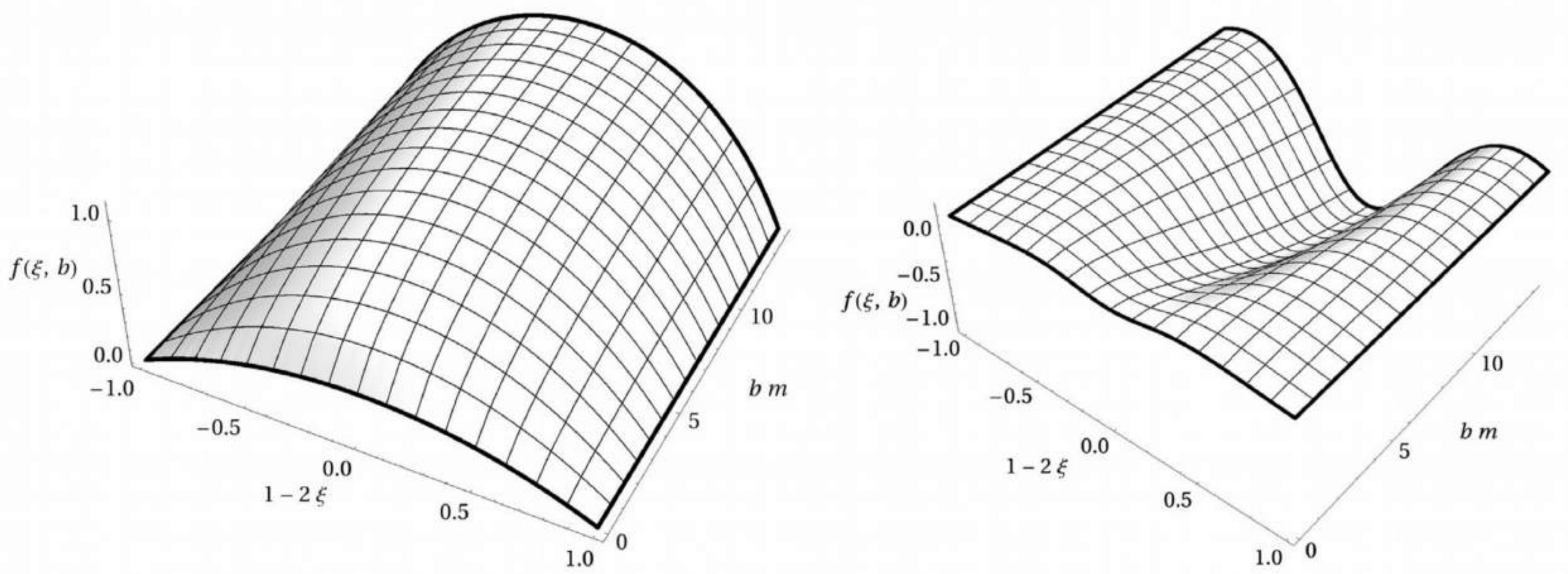


Fig. 7. The valence functions $f(\xi, b)$ in the impact parameter space. Left panel: the ground state, corresponding to $B(0) = 1.9m$, $\mu = 0.1m$ and $\alpha_{gr} = 6.437$. Right panel: first-excited state, corresponding to $B(1) = 0.22m$, $\mu = 0.1m$ and $\alpha_{gr} = 6.437$.

Gutierrez, Gigante, TF, Salmè, Viviani, Tomio PLB759 (2016) 131

Light-front valence wave function L+XL

Large momentum behavior

$$\psi_{LF}(\gamma, \xi) \to \alpha\, \gamma^{-2}\, C(\xi)$$

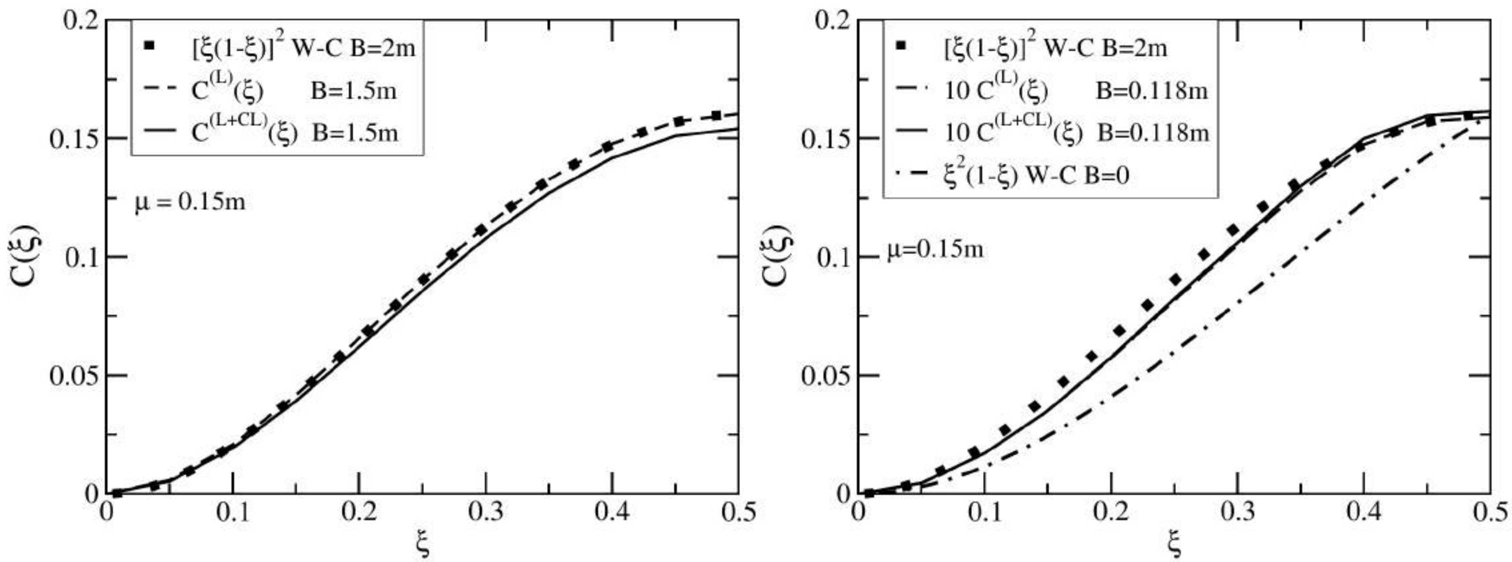


Fig. 2. Asymptotic function $C(\xi)$ defined from the LF wave function for $\gamma \to \infty$ (6) computed for the ladder kernel, $C^{(L)}(\xi)$ (dashed line), and ladder plus cross-ladder kernel, $C^{(L+CL)}(\xi)$ (solid line), with exchanged boson mass of $\mu = 0.15\,m$. Calculations are performed for $B = 1.5\,m$ (left frame) and $B = 0.118\,m$ (right frame). A comparison with the analytical forms of $C(\xi)$ valid for the Wick-Cutkosky model for $B = 2m$ (full box) and $B \to 0$ (dash-dotted line) both arbitrarily normalized.

Gigante, Nogueira, Ydrefors, Gutierrez, Karmanov, TF, PRD95(2017)056012.

Euclidean space: Nakanishi representation

Euclidean space (after the replacement $k_0 = ik_4$)

$$\boxed{\Phi_E(k_v, k_4) = \int_{-1}^{1} dz' \int_0^{\infty} d\gamma' \frac{g(\gamma', z')}{(\gamma' + k_4^2 + k_v^2 + \kappa^2 - iMk_4 z')^3}}$$

$k = (k_0, \vec{k})$ $\qquad \kappa^2 = m^2 - \frac{M^2}{4}$

- Note: Wick-rotation is the exact analytical continuation of the Minkowski space Nakanishi representation of the BS amplitude!

Transverse distribution: Euclidean and Minkowski

$$\phi_M^T(\mathbf{k}_\perp) \equiv \int dk^0 dk^3 \Phi(k,p) = \frac{1}{2}\int dk^+ dk^- \Phi(k,p) \text{ and}$$

$$\phi_E^T(\mathbf{k}_\perp) \equiv \mathrm{i}\int dk_E^0 dk^3 \Phi_E(k_E,p),$$

136 C. Gutierrez et al. / Physics Letters B 759 (2016) 131–137

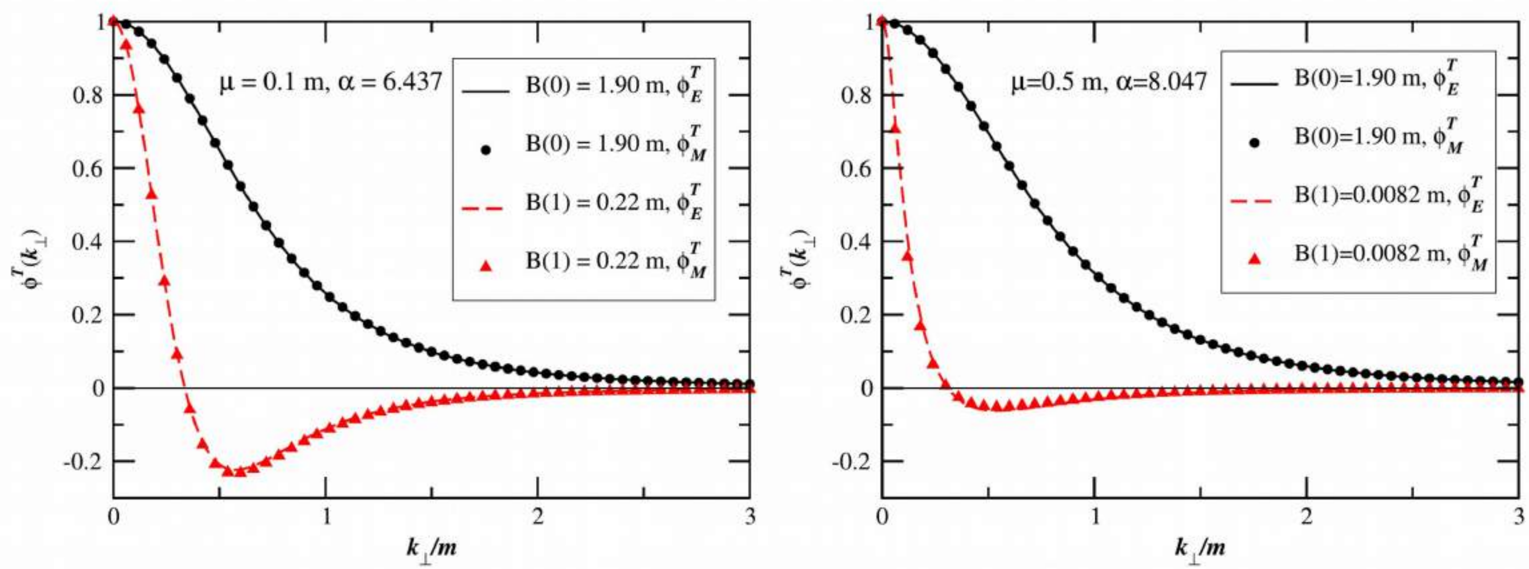


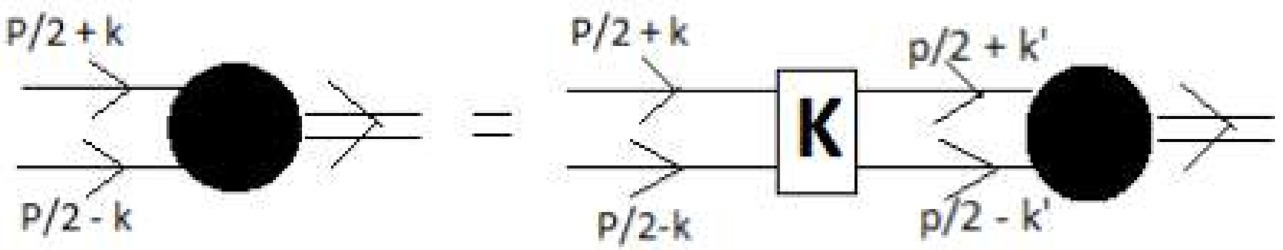
Fig. 6. Transverse momentum amplitudes *s*-wave states, in Euclidean and Minkowski spaces, vs $k_\perp$, for both ground- and first-excited states, and two values of μ/m and α_{gr} (as indicated in the insets). The amplitudes ϕ_E^T and ϕ_M^T, arbitrarily normalized to 1 at the origin, are not easily distinguishable.

Gutierrez, Gigante, TF, Salmè, Viviani, Tomio PLB759 (2016) 131

BSE for qqbar: pion

Carbonell and Karmanov EPJA 46 (2010) 387;

de Paula, TF,Salmè, Viviani PRD 94 (2016) 071901;



$$\Phi(k,p) = S(k+p/2) \int \frac{d^4k'}{(2\pi)^4} F^2(k-k')\, i\mathcal{K}(k,k')\, \Gamma_1\, \Phi(k',p)\, \bar{\Gamma}_2\, S(k-p/2)$$

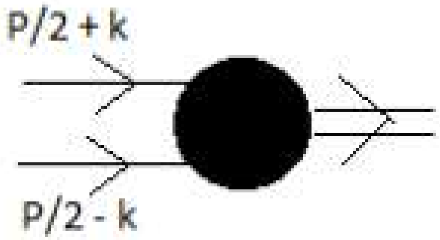
Ladder approximation (L): suppression of XL for Nc=3 [A. Nogueira, CR Ji, Ydrefors, TF, PLB(2017) 1710.04398 [hep-th]]

Vector $$i\mathcal{K}_V^{(Ld)\mu\nu}(k,k') = -ig^2 \frac{g^{\mu\nu}}{(k-k')^2 - \mu^2 + i\epsilon}$$

Vertex Form-Factor $$F(q) = \frac{\mu^2 - \Lambda^2}{q^2 - \Lambda^2 + i\epsilon}$$

NIR for fermion-antifermion: 0⁻ (pion)

BS amplitude



$$\Phi(k,p) = S_1\phi_1 + S_2\phi_2 + S_3\phi_3 + S_4\phi_4$$

$$S_1 = \gamma_5 \quad S_2 = \frac{1}{M}\not{p}\gamma_5 \quad S_3 = \frac{k\cdot p}{M^3}\not{p}\gamma_5 - \frac{1}{M}\not{k}\gamma_5 \quad S_4 = \frac{i}{M^2}\sigma_{\mu\nu}p^\mu k^\nu \gamma_5$$

$$\phi_i(k,p) = \int_{-1}^{+1} dz' \int_0^\infty d\gamma' \frac{g_i(\gamma',z')}{(k^2 + p\cdot k\; z' + M^2/4 - m^2 - \gamma' + i\epsilon)^3}$$

Light-front projection: integration over k^- (LF singularities)

For the two-fermion BSE, singularities have generic form:

$$\mathcal{C}_j = \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} (k^-)^j \, \mathcal{S}(k^-, v, z, z', \gamma, \gamma') \qquad j = 1, 2, 3$$

with $\mathcal{S}(k^-, v, z, z', \gamma, \gamma')$ explicitly calculable

N.B., in the worst case

$$\mathcal{S}(k^-, v, z, z', \gamma, \gamma') \sim \frac{1}{[k^-]^2} \qquad \text{for} \quad k^- \to \infty$$

End-point singularities: T.M. Yan , Phys. Rev. **D 7**, 1780 (1973)

$$\mathcal{I}(\beta, y) = \int_{-\infty}^{\infty} \frac{dx}{\left[\beta x - y \mp i\epsilon\right]^2} = \pm \frac{2\pi i \; \delta(\beta)}{\left[-y \mp i\epsilon\right]}$$

→ Kernel with delta's and its derivatives!

End-point singularities– more intuitive: can be treated by the pole-dislocation method
de Melo et al. NPA631 (1998) 574C, PLB708 (2012) 87

Numerical comparison: Scalar coupling

	$\mu/m = 0.15$			$\mu/m = 0.50$		
B/m	g^2_{dFSV}(full)	g^2_{CK}		g^2_{dFSV}(full)	g^2_{CK}	g^2_E
0.01	7.844	7.813		25.327	25.23	-
0.02	10.040	10.05		29.487	29.49	-
0.04	13.675	13.69		36.183	36.19	36.19
0.05	15.336	15.35		39.178	39.19	39.18
0.10	23.122	23.12		52.817	52.82	-
0.20	38.324	38.32		78.259	78.25	-
0.40	71.060	71.07		130.177	130.7	130.3
0.50	88.964	86.95		157.419	157.4	157.5
1.00	187.855	-		295.61	-	-
1.40	254.483	-		379.48	-	-
1.80	288.31	-		421.05	-	-

First column: binding energy.
Red digits: coupling constant g^2 for $\mu/m = 0.15$ and 0.50, with the analytical treatment of the fermionic singularities (present work). -
Black digits: results for $\mu/m = 0.15$ and 0.50, with a numerical treatment of the singularities (Carbonell & Karmanov EPJA **46**, (2010) 387).
Blue digits: results in Euclidean space from Dorkin et al FBS. **42** (2008) 1.

Scalar boson exchange

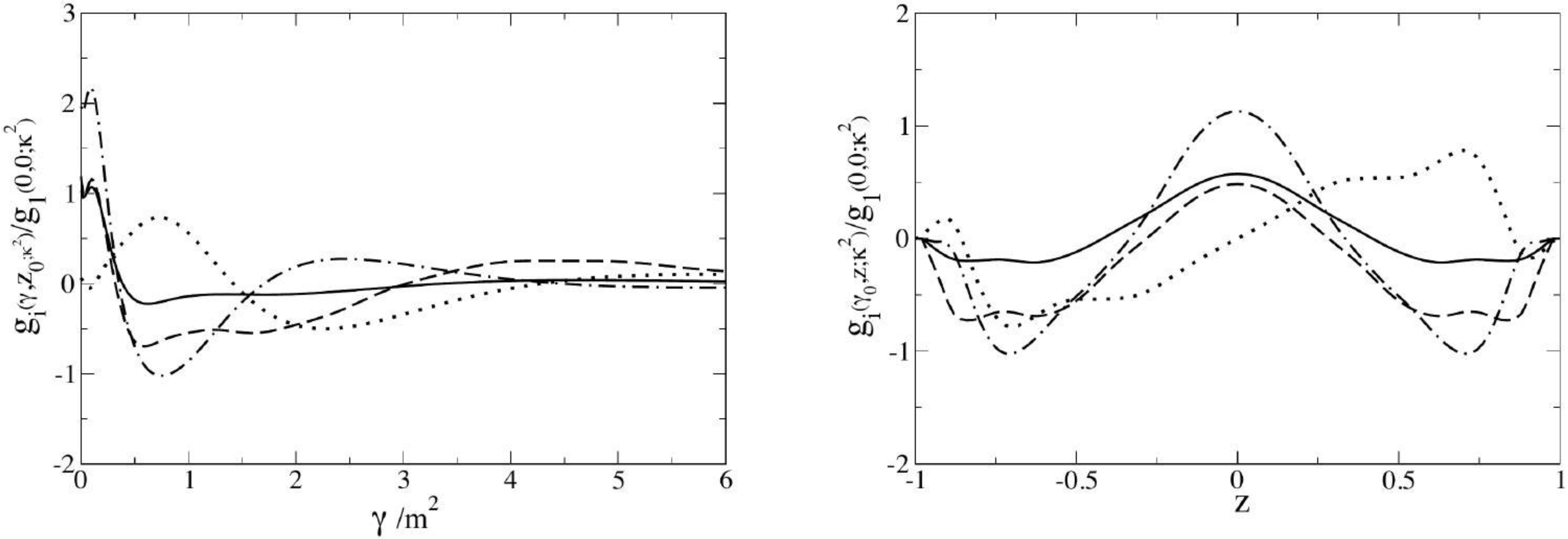
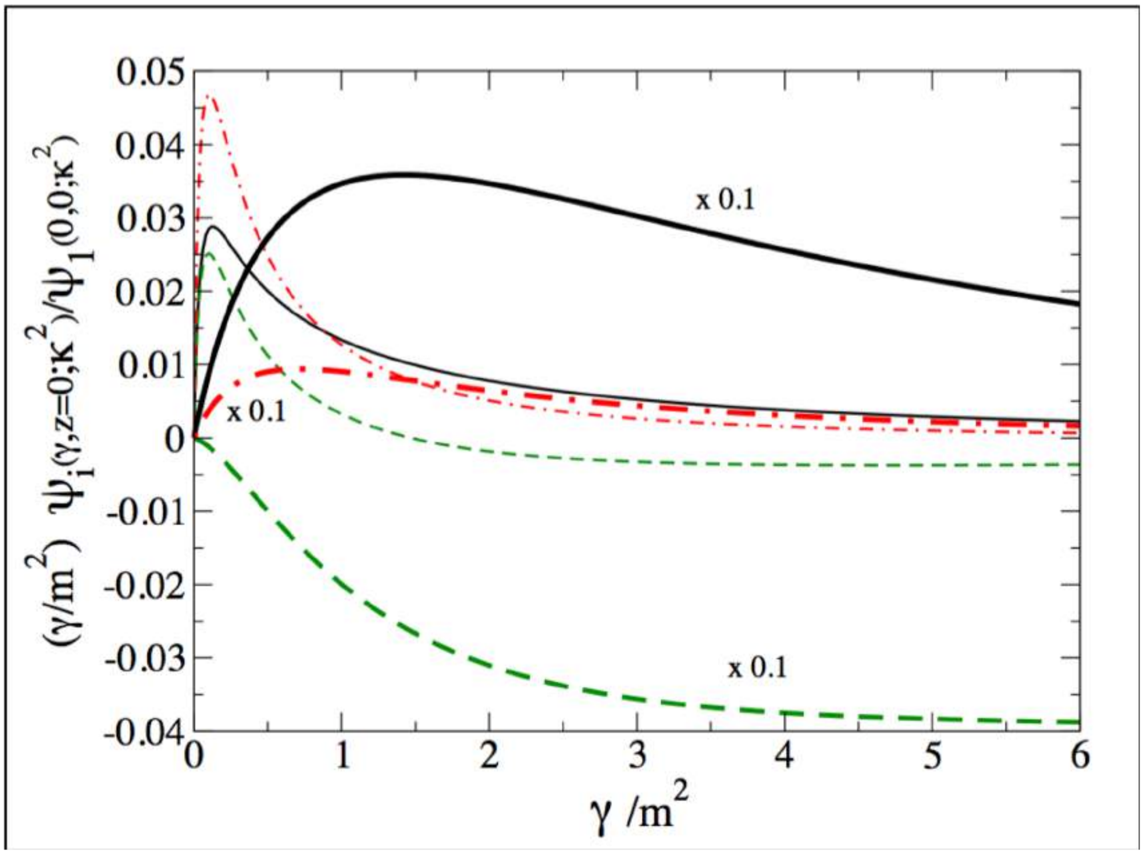


Figure 2. Nakanishi weight-functions $g_i(\gamma,\, z; \kappa^2)$, Eqs. 3.1 and 3.2 evaluated for the 0^+ two-fermion system with a scalar boson exchange such that $\mu/m = 0.5$ and $B/m = 0.1$ (the corresponding coupling is $g^2 = 52.817$ [17]). The vertex form-factor cutoff is $\Lambda/m = 2$. Left panel: $g_i(\gamma,\, z_0; \kappa^2)$ with $z_0 = 0.6$ and running γ/m^2. Right panel: $g_i(\gamma_0,\, z; \kappa^2)$ with $\gamma_0/m^2 = 0.54$ and running z, The Nakanishi weight-functions are normalized with respect to $g_1(0,0;\kappa^2)$. Solid line: g_1. Dashed line: g_2. Dotted line: g_3. Dot-dashed line: g_4.

de Paula, TF,Salmè, Viviani PRD 94 (2016) 071901;

Massless vector exchange: high-momentum tails

de Paula, TF,Salmè, Viviani PRD 94 (2016) 071901;



LF amplitudes ψ_i times γ/m^2 at fixed $z = 0$, for the vector coupling.
$B/m = 0.1$ (thin lines) and 1.0 (thick lines).

—— : $(\gamma/m^2)\ \psi_1$.
— —: $(\gamma/m^2)\ \psi_2$.
— • : $(\gamma/m^2)\ \psi_4$.
$\psi_3 = 0$ for $z = 0$

Power one is expected for the pion valence amplitude:

X Ji et al, PRL 90 (2003) 241601.

PION MODEL

W. de Paula, TF, Pimentel, Salmè, Viviani, EPJC 77 (2017) 764

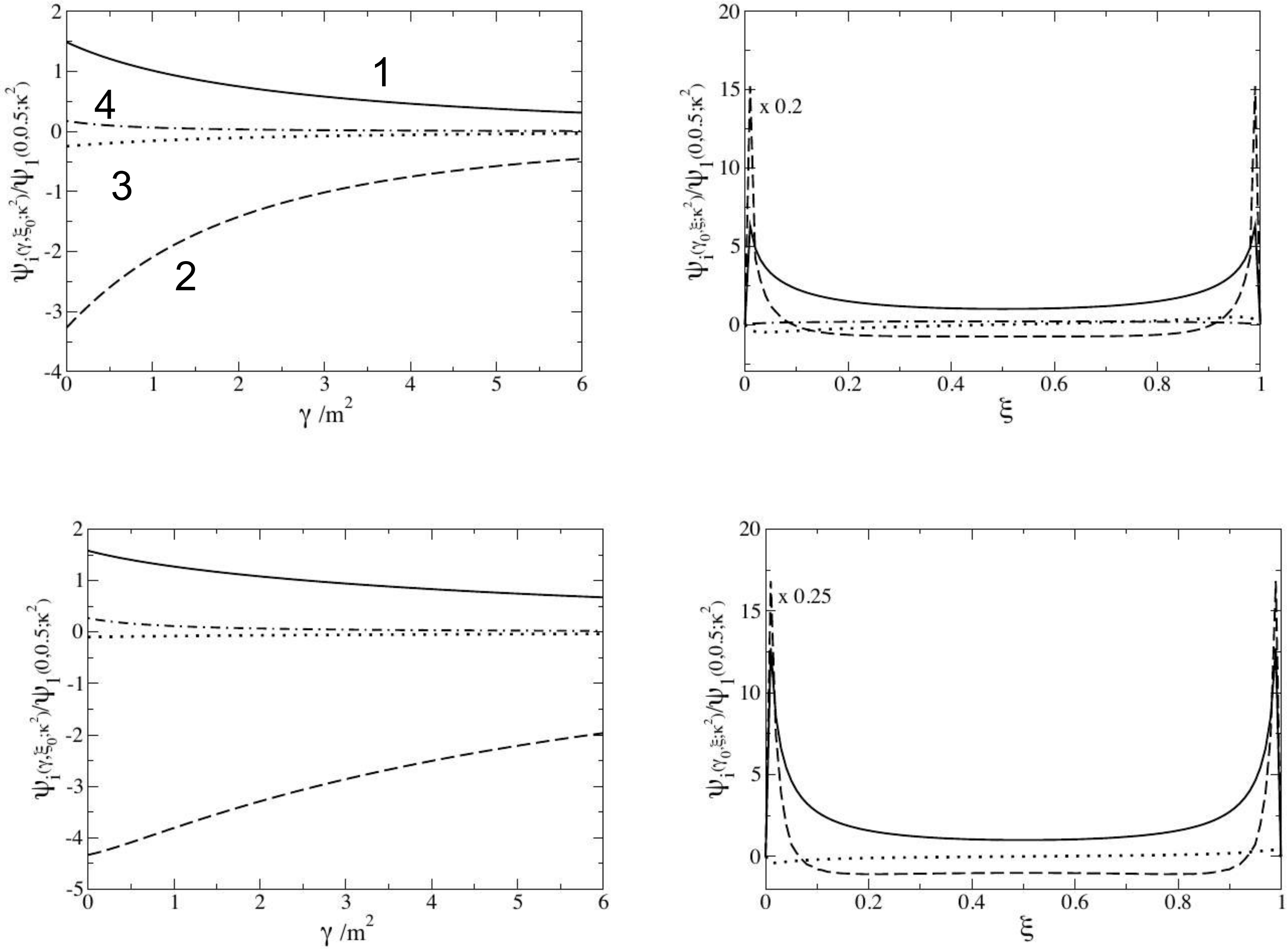
- **Gluon effective mass ~ 500 MeV – Landau Gauge LQCD**

[Oliveira, Bicudo, JPG 38 (2011) 045003;
Duarte, Oliveira, Silva, Phys. Rev. D 94 (2016) 01450240]

- **Mquark = 250 MeV**

[Parappilly, et al, PR D73 (2006) 054504]

- **Λ/m =1, 2, 3**



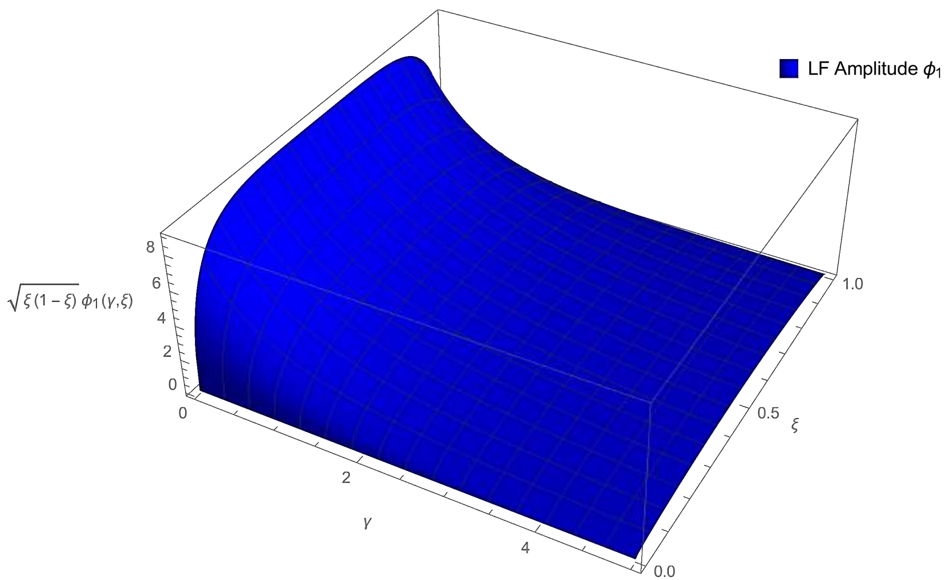
$$f_\pi = 150\,\mathrm{MeV}$$

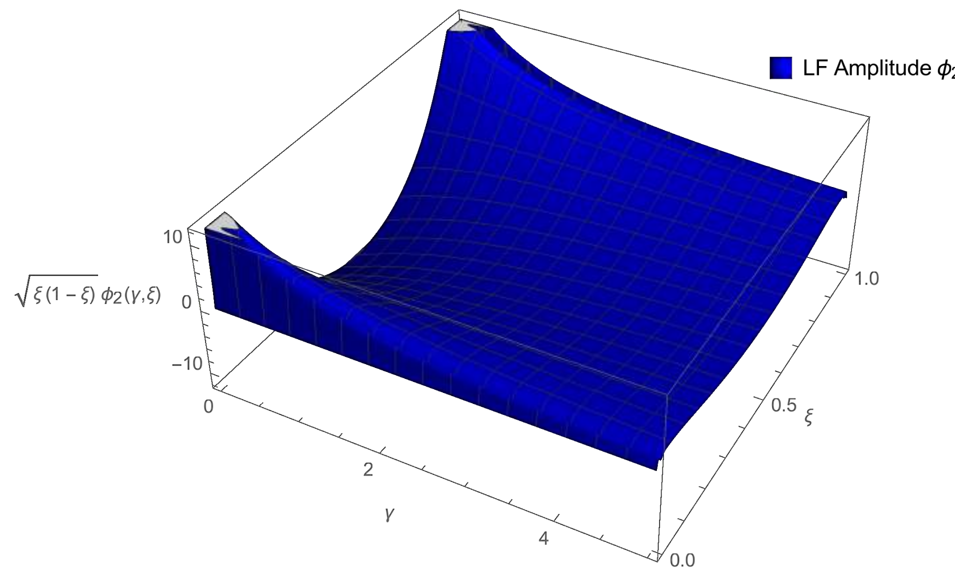
Figure 6. Light-front amplitudes $\psi_i(\gamma, \zeta)$, Eq. 3.11, for the pion-like system with a heavy-vector exchange ($\mu/m = 2$), binding energy of $B/m = 1.44$ and constituent mass $m = 250$ MeV. Upper panel: vertex form-factor cutoff $\Lambda/m = 3$ and $g^2 = 435.0$, corresponding to $\alpha_s = 10.68$ (see text for the definition of α_s). Lower panel: vertex form-factor cutoff $\Lambda/m = 8$ and $g^2 = 53.0$, corresponding to $\alpha_s = 3.71$. The value of the longitudinal variable is $\xi_0 = 0.2$ and $\gamma_0 = 0$. Solid line: ψ_1 . Dashed line: ψ_2. Dotted line: ψ_3. Dot-dashed line: ψ_4

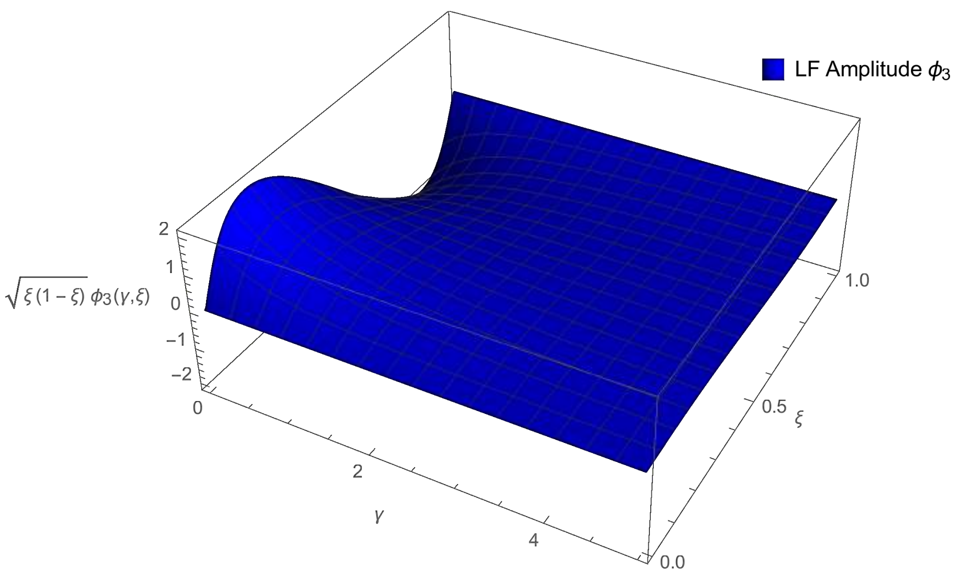
Light-front amplitudes

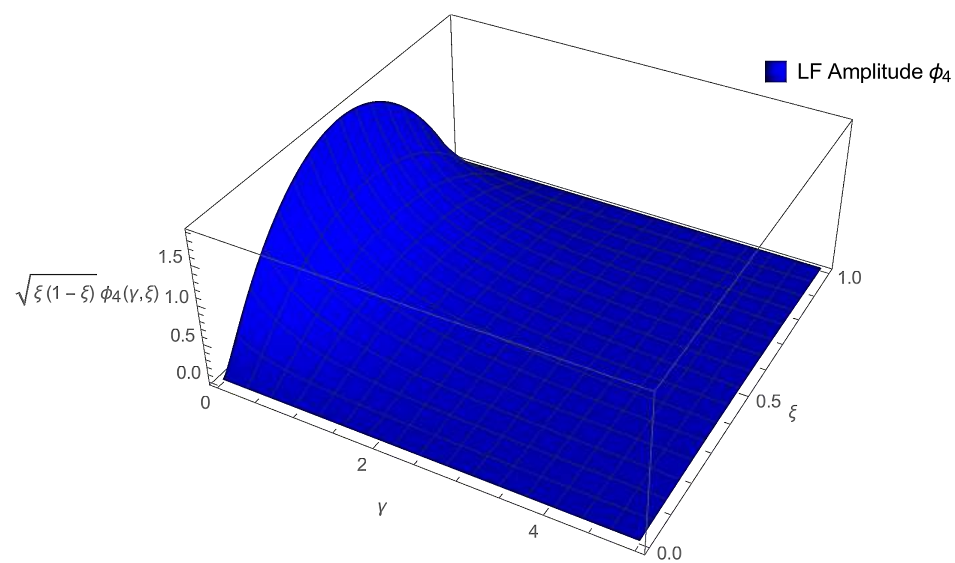
$(B/m = 1.35, \mu/m = 2.0, \Lambda/m = 1.0, \bar{m}_q$=215 MeV): $f_\pi = 96$ MeV

$P_{val} = 0.68$









Valence distribution functions

W. de Paula, et. al, in preparation

Valence probability:

$$N_2 = \frac{1}{32\,\pi^2}\int_{-1}^{1} dz \int_0^\infty d\gamma \left\{\tilde{\psi}_{val}(\gamma,\xi)\ \tilde{\psi}_{val}(\gamma,\xi) + \frac{\gamma}{M^2}\ \psi_{val;4}(\gamma,\xi)\ \psi_{val;4}(\gamma,\xi)\right\}$$

$$\tilde{\psi}_{val}(\gamma,z) = -\frac{i}{M}\int_0^\infty d\gamma' \frac{g_2(\gamma',z)}{[\gamma+\gamma'+m^2z^2+(1-z^2)\kappa^2-i\epsilon]^2}$$
$$-\frac{i}{M}\frac{z}{2}\int_0^\infty d\gamma' \frac{g_3(\gamma',z)}{[\gamma+\gamma'+m^2z^2+(1-z^2)\kappa^2-i\epsilon]^2}$$
$$+\frac{i}{M^3}\int_0^\infty d\gamma' \quad \frac{\partial g_3(\gamma',z)/\partial z}{[\gamma+\gamma'+z^2m^2+(1-z^2)\kappa^2-i\epsilon]}$$

$$\psi_{val;4}(\gamma,z) = -\frac{i}{M}\int_0^\infty d\gamma' \frac{g_4(\gamma',z)}{[\gamma+\gamma'+m^2z^2+(1-z^2)\kappa^2-i\epsilon]^2}\,.$$

Valence probability

Table 1 Valence probability for a massive vector exchange, with $\mu/m = 0.15$ and a cut-off $\Lambda/m = 2$ for the vertex form-factor. The number of gaussian points is 72.

B/m	Prob.
0.01	0.96
0.1	0.78
1.0	0.68

Table 2 Valence probability for a massive vector exchange, with $\mu/m = 0.5$ and a cut-off $\Lambda/m = 2$ for the vertex form-factor. The number of gaussian points is 72.

B/m	Prob.
0.01	0.96
0.1	0.84
1.0	0.68

Lot of room for the higher LF Fock components of the wave function to manifest!

Valence distribution functions: longitudinal and transverse

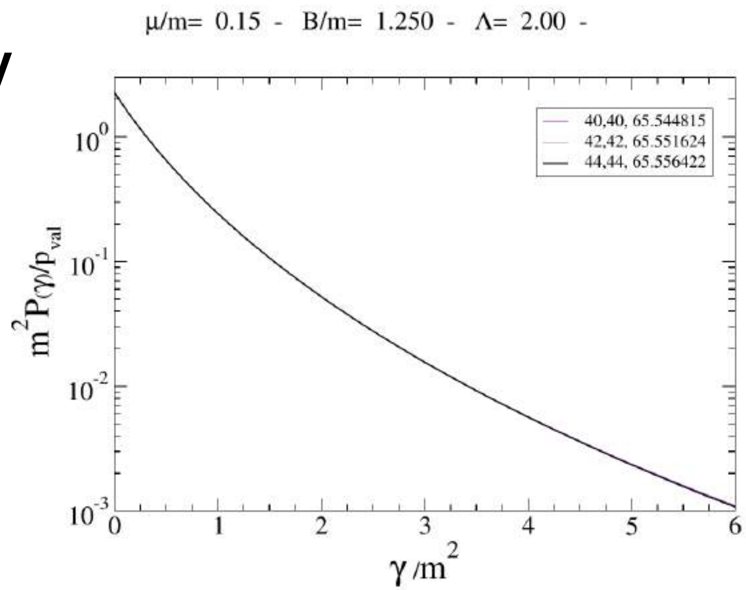
Mquark 187 MeV

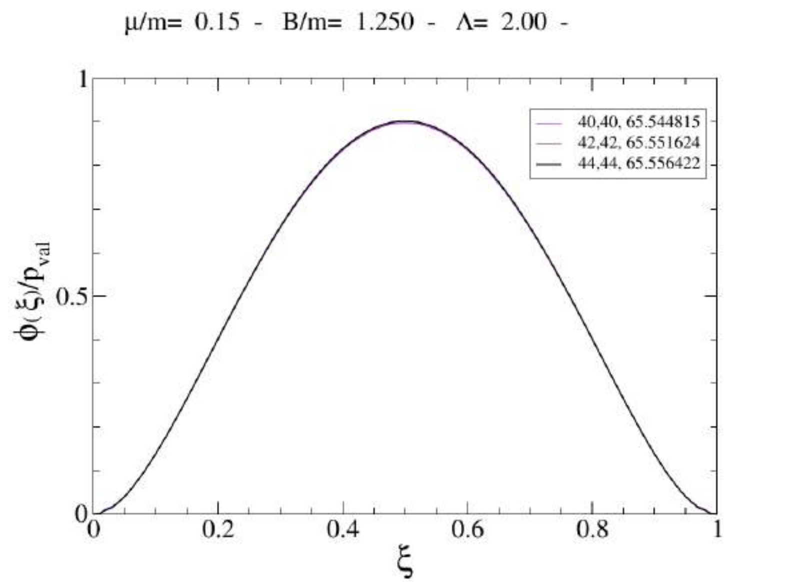
Mgluon 28 MeV

Λ/m =2

Pval=0.64

$$f_\pi = 77\,\mathrm{MeV}$$





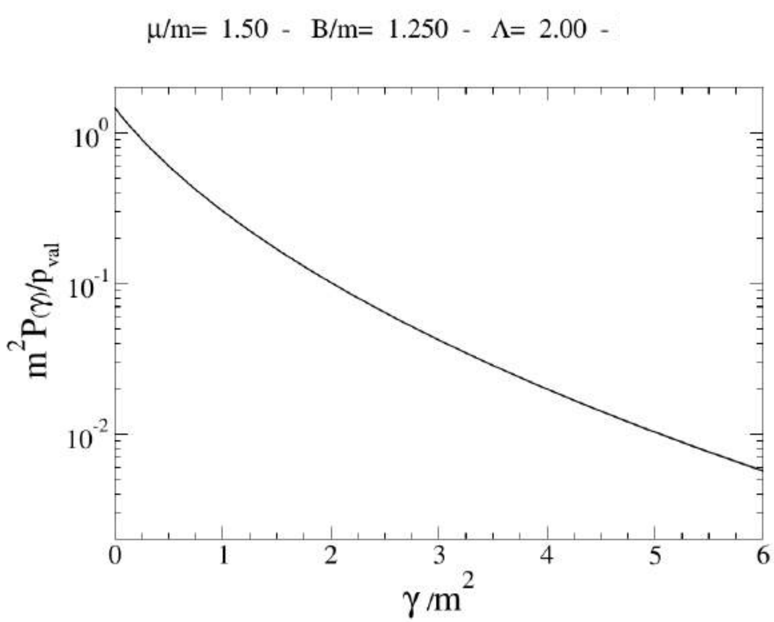
Mquark 187 MeV

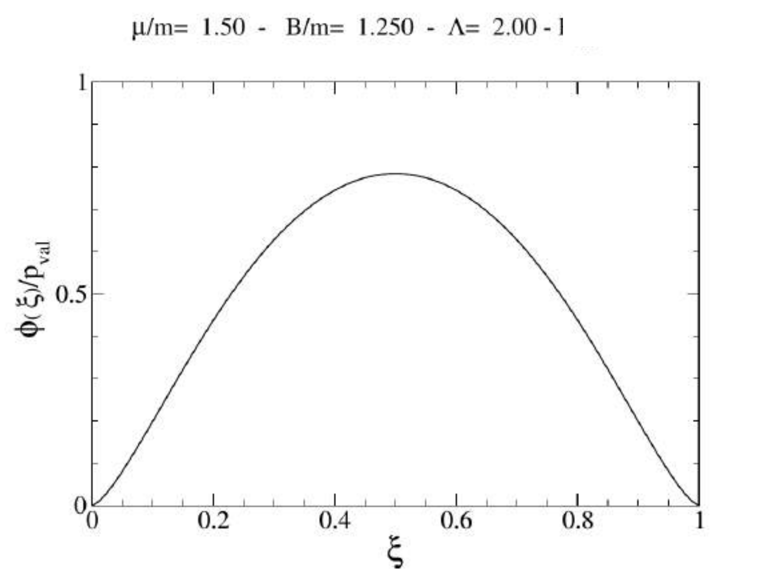
Mgluon 280 MeV

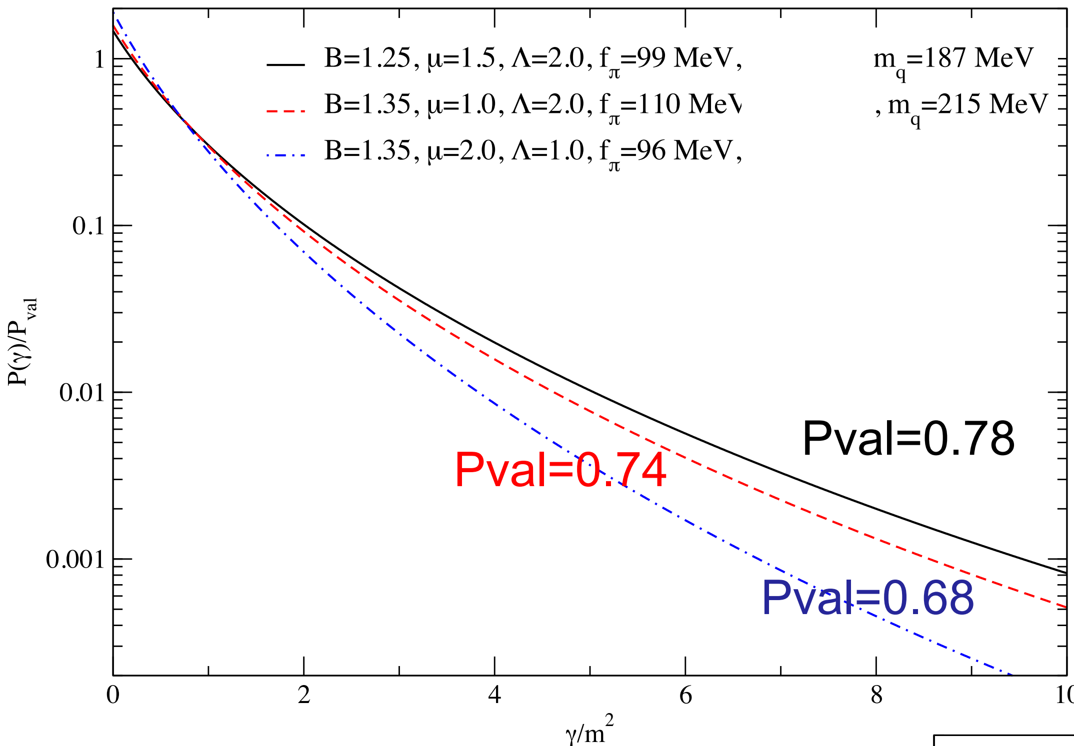
Λ/m =2

Pval=0.78

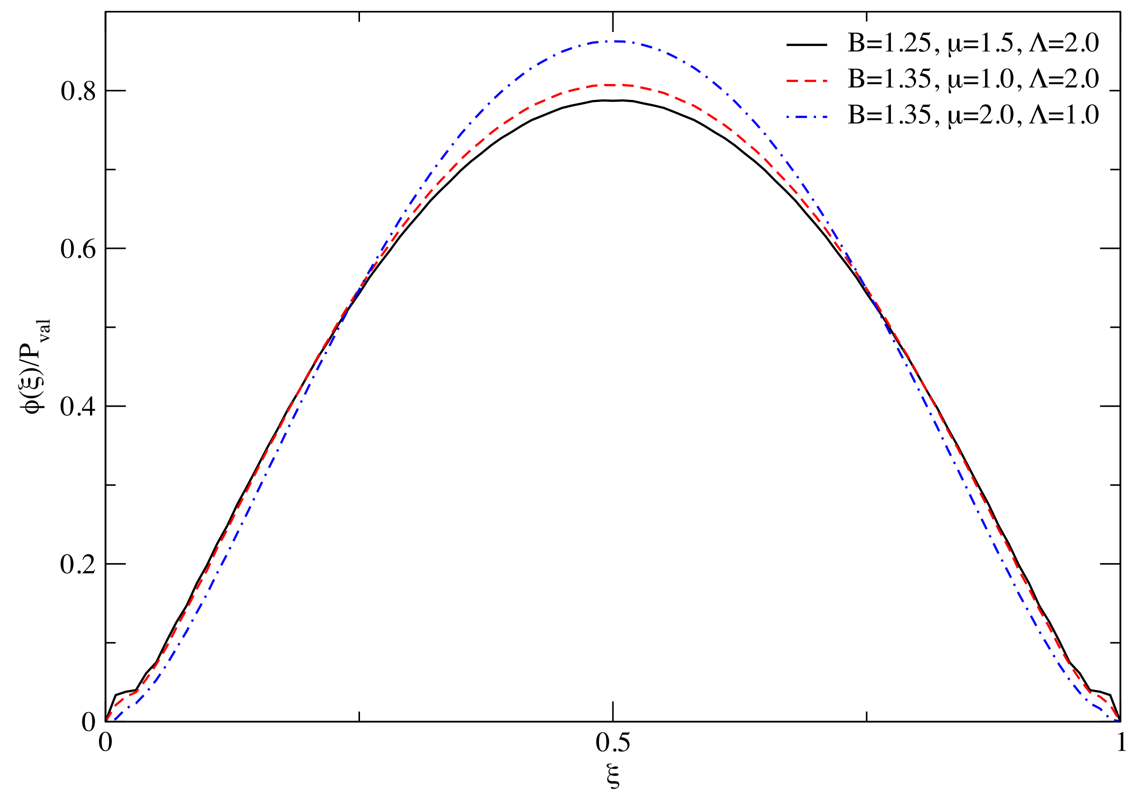
$$f_\pi = 99\,\mathrm{MeV}$$







B=1.25, μ=1.5, Λ=2.0, f_π=99 MeV,
m_q=187 MeV
B=1.35, μ=1.0, Λ=2.0, f_π=110 MeV
, m_q=215 MeV
B=1.35, μ=2.0, Λ=1.0, f_π=96 MeV,
Pval=0.78
Pval=0.74
Pval=0.68
$P(\gamma)/P_{val}$
γ/m^2



B=1.25, μ=1.5, Λ=2.0
B=1.35, μ=1.0, Λ=2.0
B=1.35, μ=2.0, Λ=1.0
$\phi(\xi)/P_{val}$
ξ

PION model with quark self-energy

C.S. Mello et al. / Physics Letters B 766 (2017) 86

The pion is the Goldston Boson of the dynamical chiral symmetry breaking
→Quark self energy and pseudo-scalar bound state equation has to be consistent in order that axial vector Ward Identity is fulfilled.

I.C. Cloët, C.D. Roberts, Prog. Part. Nucl. Phys. 77 (2014) 1.

$$\Phi(k,p) = S(\eta_1 p + k) S(\eta_2 p - k) \int \frac{d^4k'}{(2\pi)^4} iK(k,k',p)\Phi(k',p)$$

Källen–Lehmann spectral representation $\quad S(p') = \int_0^\infty d\gamma \frac{\rho(\gamma)}{p'^2 - \gamma + i\epsilon}$

$$\boxed{\Phi(k,p) = \int_0^\infty d\gamma \frac{\rho(\gamma)}{(\eta_1 p + k)^2 - \gamma + i\epsilon} \int_0^\infty d\gamma' \frac{\rho(\gamma')}{(\eta_2 p - k)^2 - \gamma' + i\epsilon} \int \frac{d^4k'}{(2\pi)^4} iK(k,k',p)\Phi(k',p)}$$

Non planar diagrams (cross-ladder) are unimportant with color degrees of freedom!

J.H. Alvarenga Nogueira, Chueng-Ryong Ji, E. Ydrefors, T. Frederico, *Phys. Lett. B 777 (2018) 207*

2. Quark model propagator

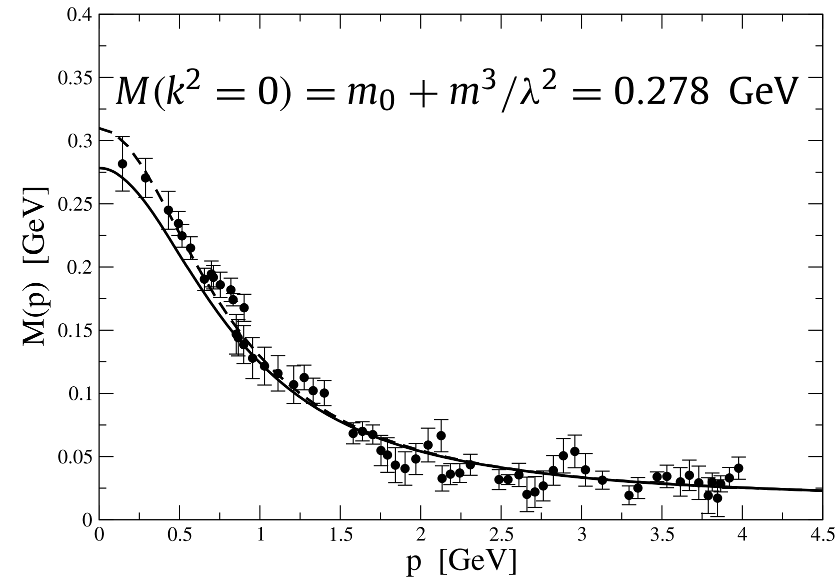


Fig. 1. The running quark mass, Eq. (3), as a function of the Euclidean momentum $p=\sqrt{-p^\mu p_\mu}$, with parameters from (4), is given by the solid line and compared to lattice QCD calculations from [37]. The dashed line shows the parametrization used in reference [51].

$$S_F(k)=\imath\, Z(k^2)\left[\not k - M(k^2)+\imath\epsilon\right]^{-1}$$

simplification $\quad Z(k^2)=1$

$$M(k^2)=m_0-m^3\left[k^2-\lambda^2+i\epsilon\right]^{-1}$$

$m_0=0.014$ GeV, $m=0.574$ GeV and $\lambda=0.846$ GeV

quark propagator poles $\begin{cases} m_i^2=M^2(m_i^2) \\ m_i\left(m_i^2-\lambda^2\right)=\pm\left[m_0\left(m_i^2-\lambda^2\right)-m^3\right] \end{cases}$

$$\boxed{S_F(k)=\imath\,\frac{(k^2-\lambda^2)^2\,(\not k+m_0)-\left(k^2-\lambda^2\right)m^3}{\prod_{i=1,3}(k^2-m_i^2+\imath\epsilon)}}$$

$m_1=0.327$ GeV, $m_2=0.644$ GeV and $m_3=0.954$ GeV

Källén–Lehmann spectral decomposition

$$S_F(k) = \imath \left[A(k^2)\, \not{k} + B(k^2) \right]$$

$$A(k^2) = \int_0^\infty d\mu^2 \frac{\rho_A(\mu^2)}{k^2 - \mu^2 + \imath\varepsilon} \quad \text{and} \quad B(k^2) = \int_0^\infty d\mu^2 \frac{\rho_B(\mu^2)}{k^2 - \mu^2 + \imath\varepsilon}$$

$$\rho_A(\mu^2) = -\frac{1}{\pi}\,\mathrm{Im}\,[A(\mu^2)] \quad \text{and} \quad \rho_B(\mu^2) = -\frac{1}{\pi}\,\mathrm{Im}\,[B(\mu^2)]$$

positivity constraints

$$\boxed{\mathcal{P}_a = \rho_A(\mu^2) \geq 0 \quad \text{and} \quad \ldots = \mu\, \rho_A(\mu^2) - \rho_B(\mu^2) \geq 0}$$

Positivity Violation!

3. Pion Bethe–Salpeter amplitude model

pion-quark-antiquark vertex

$$\Gamma_\pi(k;P) = \gamma_5[\imath E_\pi(k;P) + \not{P} F_\pi(k;P) + k^\mu P_\mu \not{k} G_\pi(k;P) + \sigma_{\mu\nu} k^\mu P^\nu H_\pi(k;P)]$$

chiral limit $m_\pi = 0$ axial-vector Ward–Takahashi identity

I.C. Cloët, C.D. Roberts, Prog. Part. Nucl. Phys. 77 (2014) 1.

$$\boxed{f_\pi E_\pi(k,P) = M(k^2)/\sqrt{Z(k^2)}}$$

OUR MODEL:

$$\boxed{\Gamma_\pi(k;P) = \imath \mathcal{N} \gamma_5 M(k^2)|_{m_0=0} \quad Z(k^2) = 1}$$

$$\boxed{\psi_\pi(k;P) = S_F(k+P/2)\, \Gamma_\pi(k;P)\, S_F(k-P/2)}$$

Other examples of analytic vertex models with fixed constituent quark masses

$$\Lambda_\pi(k,P) = \mathcal{N}\left[\frac{1}{(k+\frac{P}{2})^2 - m_R^2 + \imath\epsilon} + \frac{1}{(k-\frac{P}{2})^2 - m_R^2 + \imath\epsilon}\right]$$

J.P.B.C. de Melo, T. Frederico, E. Pace, G. Salmé, Nucl. Phys. A 707 (2002) 399; Braz. J. Phys. 33 (2003) 301.

$$\Lambda_\pi(k,P) = \mathcal{N}\,\frac{1}{\left[(k+\frac{P}{2})^2 - m_R^2 + \imath\epsilon\right]}\,\frac{1}{\left[(k-\frac{P}{2})^2 - m_R^2 + \imath\epsilon\right]}$$

T. Frederico, E. Pace, B. Pasquini, G. Salmè, Phys. Rev. D 80 (2009) 054021.
C. Fanelli, E. Pace, G. Romanelli, G. Salmè, M. Salmistraro, Eur. Phys. J. C 76 (2016) 253.

4. Integral representation of the BSA

$$\boxed{\psi_\pi(k;P) = -\left[A(k_q^2)\,\not{k}_q + B(k_q^2)\right]\frac{\mathcal{N}\gamma_5}{k^2-\lambda^2+\imath\epsilon}\left[A(k_{\bar{q}}^2)\,\not{k}_{\bar{q}} + B(k_{\bar{q}}^2)\right]}$$

$k_q = k + P/2$ and $k_{\bar{q}} = k - P/2$

$$\boxed{\frac{1}{((k+\frac{p}{2})^2-\mu'^2+\imath\epsilon)(k^2-\lambda^2+\imath\epsilon)(k-\frac{p}{2})^2-\mu^2+\imath\epsilon)} = \int_{-\infty}^{+\infty} d\gamma \int_{-1}^{1} dz \frac{g(\gamma,z;\mu',\mu,p)}{\left(k^2+zk\cdot p+\gamma+\imath\epsilon\right)^3}}$$

$$\boxed{\begin{aligned} &g(\gamma,z;\mu',\mu,p) = \\ &\quad = \frac{\theta(\alpha)\theta(1-\alpha)}{\frac{1}{2}-\alpha}\left[\theta(1-2\alpha-z)\theta(z)-\theta(z-1+2\alpha)\theta(-z)\right]\end{aligned}}$$

$$\alpha = \frac{\frac{p^2}{4}+\lambda^2-\mu^2-z^{-1}(\lambda^2+\gamma)}{\mu^2-\mu'^2+2z^{-1}(\lambda^2+\gamma)}$$

Nakanishi integral representation

$$\psi_\pi(k;P) = \gamma_5\,\chi_1(k,P) + \not{k}_q\,\gamma_5\,\chi_2(k,P) + \gamma_5\,\not{k}_{\bar{q}}\,\chi_3(k,P) + \not{k}_q\,\gamma_5\,\not{k}_{\bar{q}}\,\chi_4(k,P)$$

$$\chi_i(k,P) = \int_{-\infty}^{+\infty} d\gamma \int_{-1}^{1} dz \frac{g_i(\gamma,z;p)}{\left(k^2 + z k\cdot p + \gamma + \imath\epsilon\right)^3}$$

J. Carbonell, V.A. Karmanov, Eur. Phys. J. A 46 (2010) 387

W. de Paula, T. Frederico, G. Salmè , M. Viviani   Phys.Rev. D94 (2016) R071901

W. de Paula, T.Frederico, G. Salmè, M.Viviani and R.Pimentel, Eur. Phys. J.C 77 (2017) 764

The Nakanishi weight functions given by:

$$g_i(\gamma,z;p) = -\mathcal{N}\int_0^\infty d\mu^2 \int_0^\infty d\mu'^2\,\rho_{C_i'}(\mu'^2)\,\rho_{C_i}(\mu^2)\,g(\gamma,z;\mu',\mu,p)$$

$(C_1', C_1) = (B, B)$, $(C_2', C_2) = (A, B)$, $(C_3', C_3) = (B, A)$ and $(C_4', C_4) = (A, A)$

5. Electroweak decay constant

$$P^{\mu} f_{\pi} = N_c \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^{\mu}\gamma^{5}\psi_{\pi}(k;\,P)\right]$$

6. Electromagnetic form factor *Space-like*

$$\boxed{-\imath\,\Gamma^{\mu}_{\pi}(P,\,P';\,q) \equiv \left\langle \pi(P')\right| J^{\mu} \left|\pi(P)\right\rangle = (P + P')^{\mu} F_{\pi}(Q^2) \qquad Q^2 = -q^2}$$

$$\Gamma^{\mu}_{\pi^+}(P,\,P';\,q) = \hat{Q}_u \Gamma^{\mu}_{\pi^+,u}(P,\,P';\,q) + \hat{Q}_{\bar{d}} \Gamma^{\mu}_{\pi^+,\bar{d}}(P,\,P';\,q)$$

$$\boxed{\begin{aligned}&\Gamma^{\mu}_{\pi^+,u}(P,\,P';\,q)\\ &= N_c \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\big[S_F(k' - P'/2)\bar{\Gamma}_{\pi^+}(k';\,P')S_F(k' + P'/2)\\ &\quad\times \underbrace{\Gamma^{\mu}_{u}(k' + P'/2,\,k + P/2;\,P)}_{\text{Dressed quark photon vertex}} S_F(k + P/2)\Gamma_{\pi^+}(k;\,P)\big]\,,\;\; k' = k + q/2\end{aligned}}$$

7. Dressed quark current operator

$$q_\mu \Gamma^\mu_\pi (P, P'; q) = 0$$

Ward–Takahashi identity (WTI)

$$q_\mu \Gamma^\mu_q (p', p; q) = S_F^{-1}(p') - S_F^{-1}(p)$$

$$\boxed{-\imath\, \Gamma^\mu_q (p', p; q) = \gamma^\mu - \frac{m^3 (p' + p)^\mu}{\mathcal{D}(p'^2)\mathcal{D}(p^2)}}$$

$$\mathcal{D}(p^2) = \left(p^2 - \lambda^2 + i\epsilon\right) \qquad q = p' - p$$

Electromagnetic form-factor

8. Results $r_\pi = 0.672\ \text{fm},\ f_\pi = 90\ \text{MeV}$

$$r_\pi^{exp} = 0.672 \pm 0.008\ \text{fm} \quad f_\pi^{exp} = 92.42 \pm 0.021\ \text{MeV}$$

K.A. Olive, et al., Particle Data Group, Chin. Phys. C 38 (2014) 090001

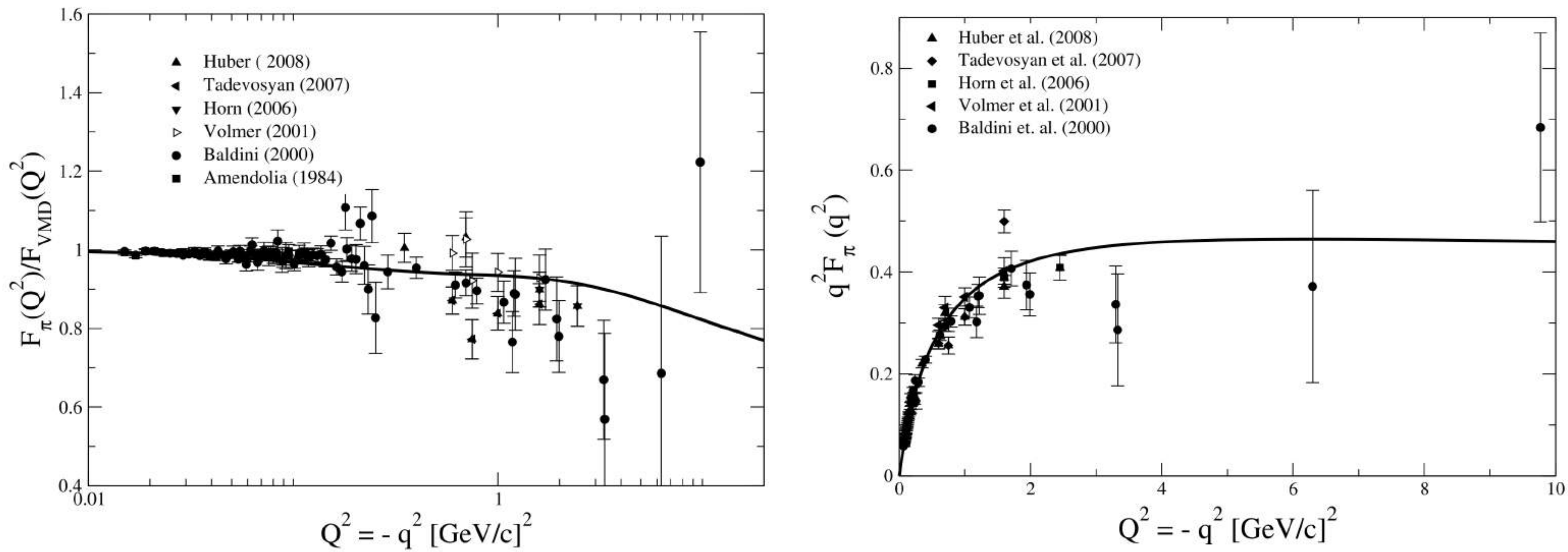


Fig. 3. Pion model electromagnetic form factor as a function of the space-like momentum transfer, $Q^2 = -q^2$, compared to the experimental values: Amendolia et al. [70], Baldini et al. [71], Volmer et al. [72], Horn et al. [73], Tadevosyan et al. [74] Huber et al. [75]. In the left frame it is presented the results normalized to the dipole form factor, $F_\pi(Q^2)(1 + Q^2/(0.77\ \text{GeV})^2)$, and in the right frame $Q^2 F_\pi(Q^2)$.

Preliminary result for a fermion-scalar bound system

The covariant decomposition of the BS amplitude for a $(1/2)^+$ bound system, composed by a fermion and a scalar, reads

$$\Phi(k,p) = \Big[S_1\ \phi_1(k,p) + S_2\ \phi_2(k,p)\Big]\ U(p,s)$$

with $U(p,s)$ a Dirac spinor, $S_1(k) = 1$, $S_2(k) = \not{k}/M$,and $M^2 = p^2$

A first check: scalar coupling $\alpha^s = \lambda^s_F \lambda^s_S/(8\pi m_S)$, for $m_F = m_S$ and $\mu/\bar{m} = 0.15,\ 0.50$

$B/\bar{m}$	$\alpha^s_M(0.15)$	$\alpha^s_{WR}(0.15)$	$\alpha^s_M(0.50)$	$\alpha^s_{WR}(0.50)$
0.10	1.5057	1.5057	2.6558	2.6558
0.20	2.2969	2.2969	3.2644	3.6244
0.30	3.0467	3.0467	4.5354	4.5354
0.40	3.7963	3.7963	5.4505	5.4506
0.50	4.5680	4.5681	6.4042	6.4043
0.80	7.2385	7.2387	9.8789	9.8794
1.00	9.7779	9.7783	13.7379	13.7380

First column: the binding energy in unit of $\bar{m} = (m_S + m_F)/2$..

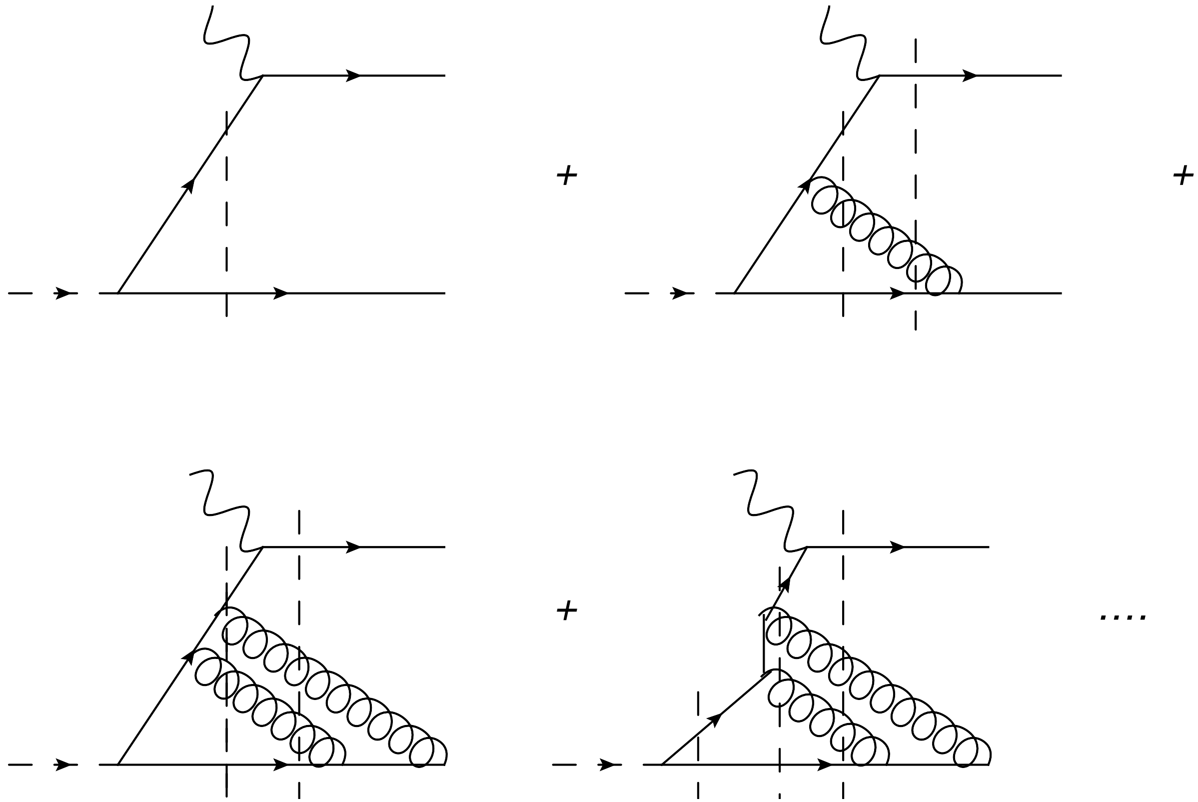
Second and fourth columns: coupling constant α_M, obtained by solving the BSE in Minkowski space, for given $B/\bar{m}$.

Third and fifth columns: Wick-rotated results, α_{WR}.

Beyond the valence ….

Sales, TF, Carlson,Sauer, PRC 63, 064003 (2001)

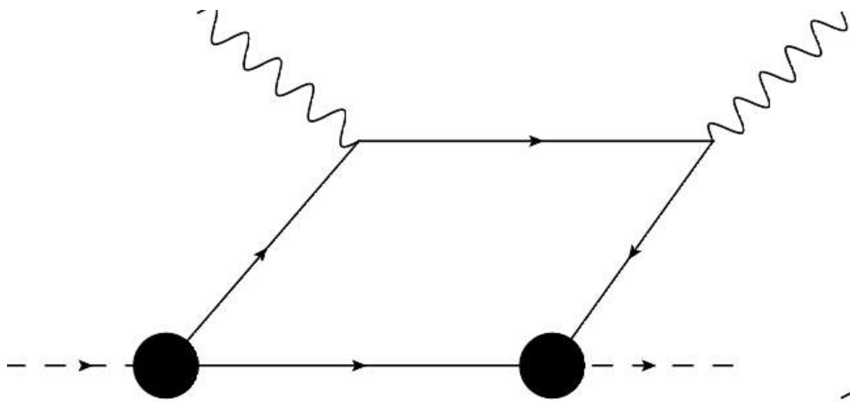
Marinho, TF, Pace,Salme,Sauer, PRD 77, 116010 (2008)

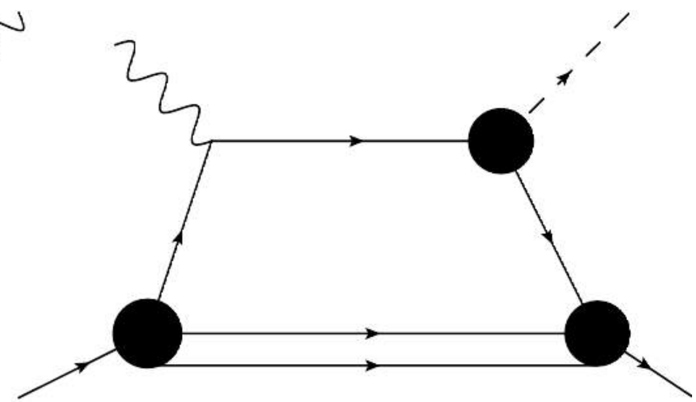


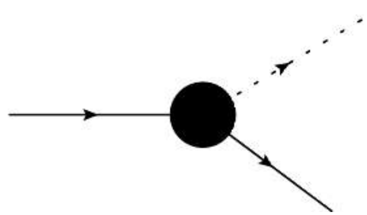
- **Population of lower x, due to the gluon radiation!**
- **Evolution?**

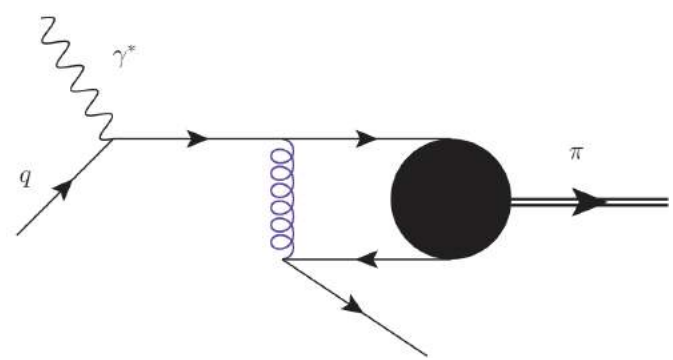
Beyond the valence ….

ERBL – DGLAP regions









Fragmentation function

Conclusions and Perspectives

- **A method for solving the fermionic BSE with Nakanishi Integral Representation (NIR): LF singularities solved for the fermionic bound state system**

- **Our numerical results confirm the robustness of the NIR for solving the BSE: Application to model the pion state: valence prob, fpi, pdf and LF amplitudes**

- **Inversion of the Nakanishi weight function from the valence…**

- **More realism: self-energies, vertex corrections, Landau gauge, ingredients from LQCD….**

- **Confinement?**

- **Beyond the pion, kaon, D, B, rho…, and the nucleon**

- **Form-Factors, PDFs, QPDs,TMDs, Fragmentation Functions...**

THANK YOU!

CAPES 65 anos

CNPq

FAPESP