Light meson form factors in light-front dynamics
with color singlet Nambu-Jona-Lasinio interactions

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BLFQ-NJL in collaboration with Wenyang Qian and J. Vary

also presenting BLFQ results from Lekha Adhikari, Guangyao Chen, Meijian Li, Pieter Maris, Shuo Tang, Anji Yu, Chandan Mondal, and Xingbo Zhao

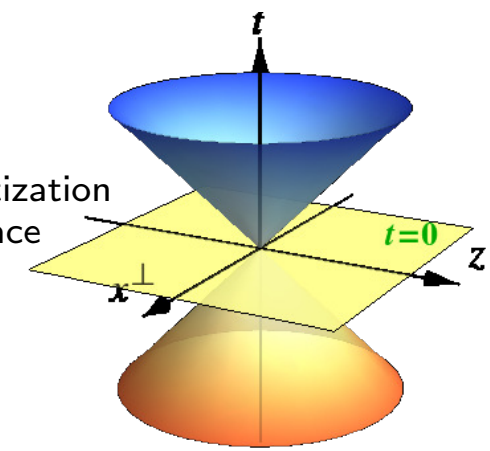
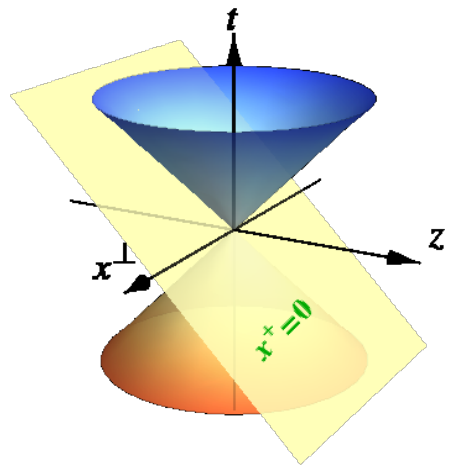
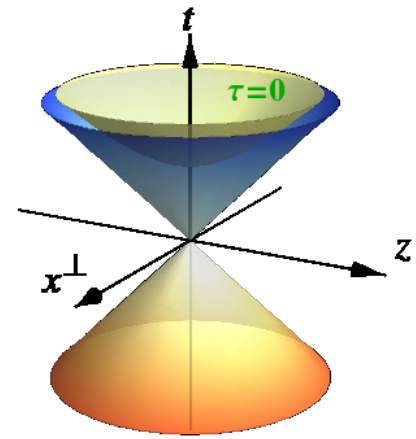
Workshop on Pion and Kaon Structure at an Electron-Ion Collider
May 28, 2018
Catholic University of America

Recent progress on the basis light-front quantization approach to hadron physics

The basis light-front quantization - Nambu-Jona-Lasinio model for the light mesons

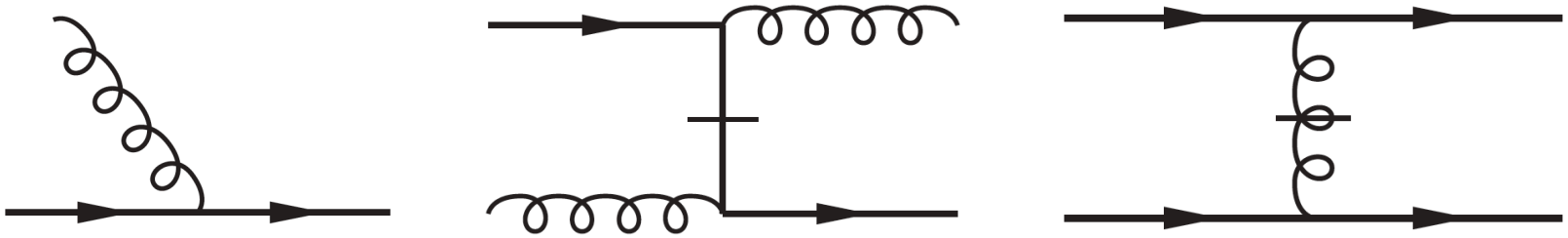
Dirac's forms of relativistic dynamics [Dirac, Rev. Mod. Phys. **21**, 392 1949]
Instant form is the well-known form of dynamics starting with $x^0 = t = 0$
$K^i = M^{0i}$, $J^i = \frac{1}{2}\varepsilon^{ijk}M^{jk}$, $\varepsilon^{ijk} = (+1,-1,0)$ for (cyclic, anti-cyclic, repeated) indeces
Front form defines relativistic dynamics on the light front (LF): $x^+ = x^0+x^3 = t+z = 0$

$P^\pm \triangleq P^0 \pm P^3$, $\vec{P}^\perp \triangleq (P^1, P^2)$, $x^\pm \triangleq x^0 \pm x^3$, $\vec{x}^\perp \triangleq (x^1, x^2)$, $E^i = M^{+i}$, $E^+ = M^{+-}$, $F^i = M^{-i}$

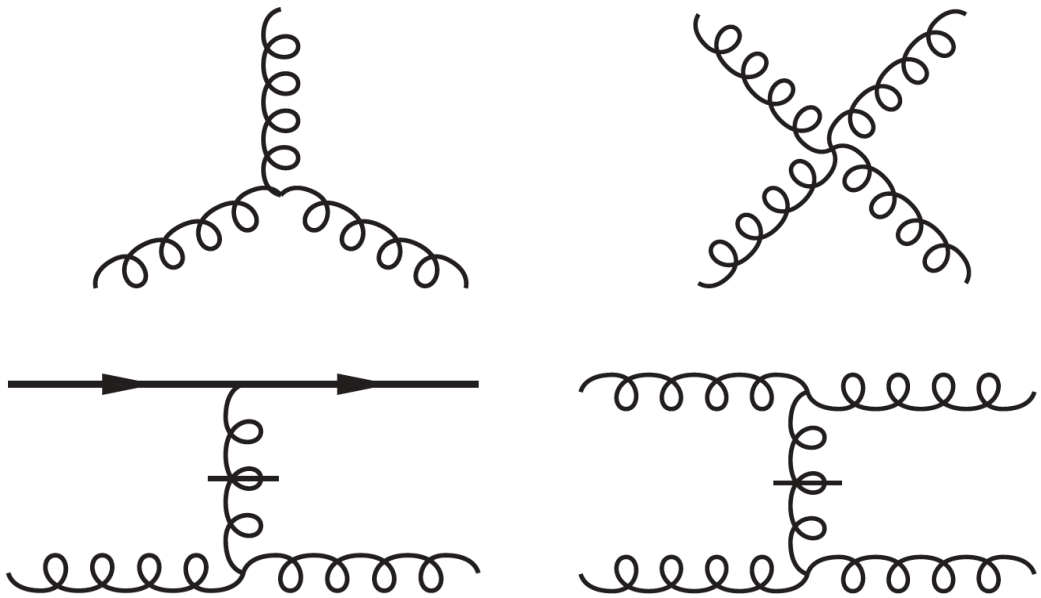
	instant form	front form	point form
time variable	$t = x^0$	$x^+ \triangleq x^0 + x^3$	$\tau \triangleq \sqrt{t^2 - \vec{x}^2 - a^2}$
quantization surface			
Hamiltonian	$H = P^0$	$P^- \triangleq P^0 - P^3$	P^μ
kinematical	$\vec{P}, \vec{J}$	$\vec{P}^\perp, P^+, \vec{E}^\perp, E^+, J^-$	$\vec{J}, \vec{K}$
dynamical	$\vec{K}, P^0$	$\vec{F}^\perp, P^-$	$\vec{P}, P^0$
dispersion relation	$p^0 = \sqrt{\vec{p}^2 + m^2}$	$p^- = (\vec{p}_\perp^2 + m^2)/p^+$	$p^\mu = mv^\mu \ (v^2 = 1)$

Adapted from talk by Yang Li

Light Front (LF) Hamiltonian Defined by its Elementary Vertices in LF Gauge



QED & QCD



QCD

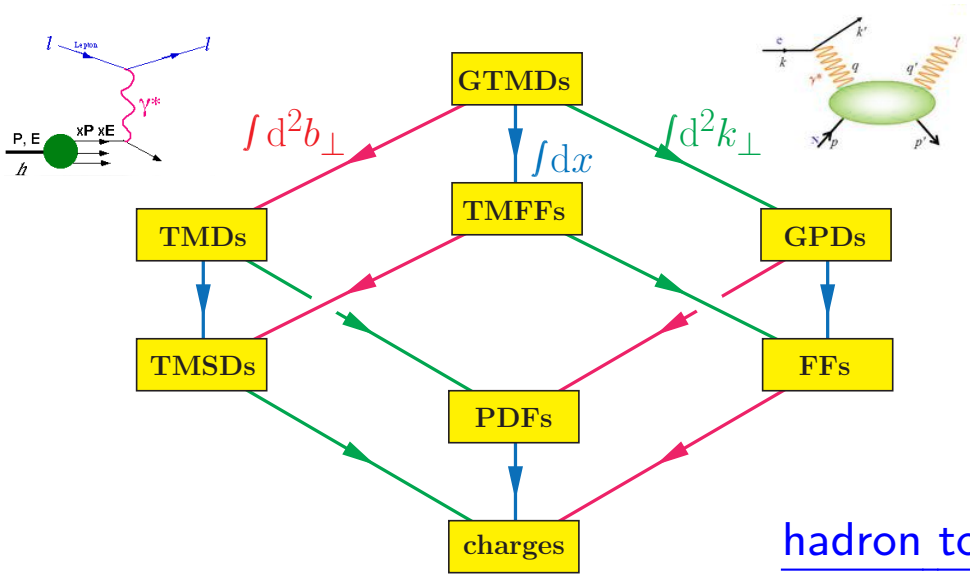
Light-Front Wavefunctions (LFWFs)

$$|\psi_h(P,j,\lambda)\rangle = \sum_n \int [\mathrm{d}\mu_n]\, \psi_{n/h}(\{\vec{k}_{i\perp}, x_i, \lambda_i\}_n)|\{\vec{p}_{i\perp}, p_i^+, \lambda_i\}_n\rangle$$

LFWFs are *frame-independent* (boost invariant) and depend only on the relative variables: $x_i \equiv p_i^+/P^+, \vec{k}_{i\perp} \equiv \vec{p}_{i\perp} - x_i\vec{P}_\perp$

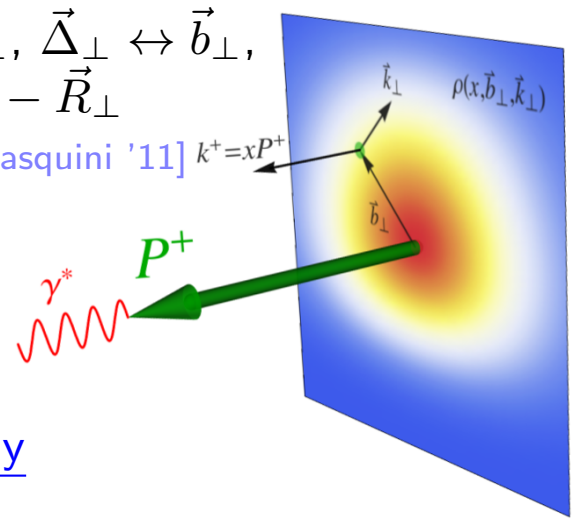
LFWFs provide intrinsic information of the structure of hadrons, and are indispensable for exclusive processes in DIS [Lepage '80]

- Overlap of LFWFs: structure functions (e.g. PDFs), form factors, ...
- Integrating out LFWFs: light-cone distributions (e.g. DAs)



$\vec{k}_\perp \leftrightarrow \vec{r}_\perp,\ \vec{\Delta}_\perp \leftrightarrow \vec{b}_\perp,$
$\vec{b}_\perp = \vec{r}_\perp - \vec{R}_\perp$

[Lorcé & Pasquini '11]



hadron tomography

Discretized Light Cone Quantization

[H.C. Pauli & S.J. Brodsky, PRD32 (1985)]

⇩

Basis Light Front Quantization

[J.P. Vary, et al., PRC81 (2010)]

$$\phi\left(\vec{k}_\perp, x\right) = \sum_\alpha \left[f_\alpha\left(\vec{k}_\perp, x\right) a_\alpha + f_\alpha^*\left(\vec{k}_\perp, x\right) a_\alpha^\dagger \right]$$

where $\{a_\alpha\}$ satisfy usual (anti-) commutation rules.

Furthermore, $f_\alpha\left(\vec{x}\right)$ are arbitrary except for conditions:

Orthonormal: $$\int f_\alpha\left(\vec{k}_\perp, x\right) f_{\alpha'}^*\left(\vec{k}_\perp, x\right) \frac{d^2 k_\perp dx}{(2\pi)^3 2x(1-x)} = \delta_{\alpha\alpha'}$$

Complete: $$\sum_\alpha f_\alpha\left(\vec{k}_\perp, x\right) f_\alpha^*\left(\vec{k}'_\perp, x'\right) = 16\pi^3 \sqrt{x(1-x)}\, \delta^2\left(\vec{k}_\perp - \vec{k}'_\perp\right) \delta\left(x - x'\right)$$

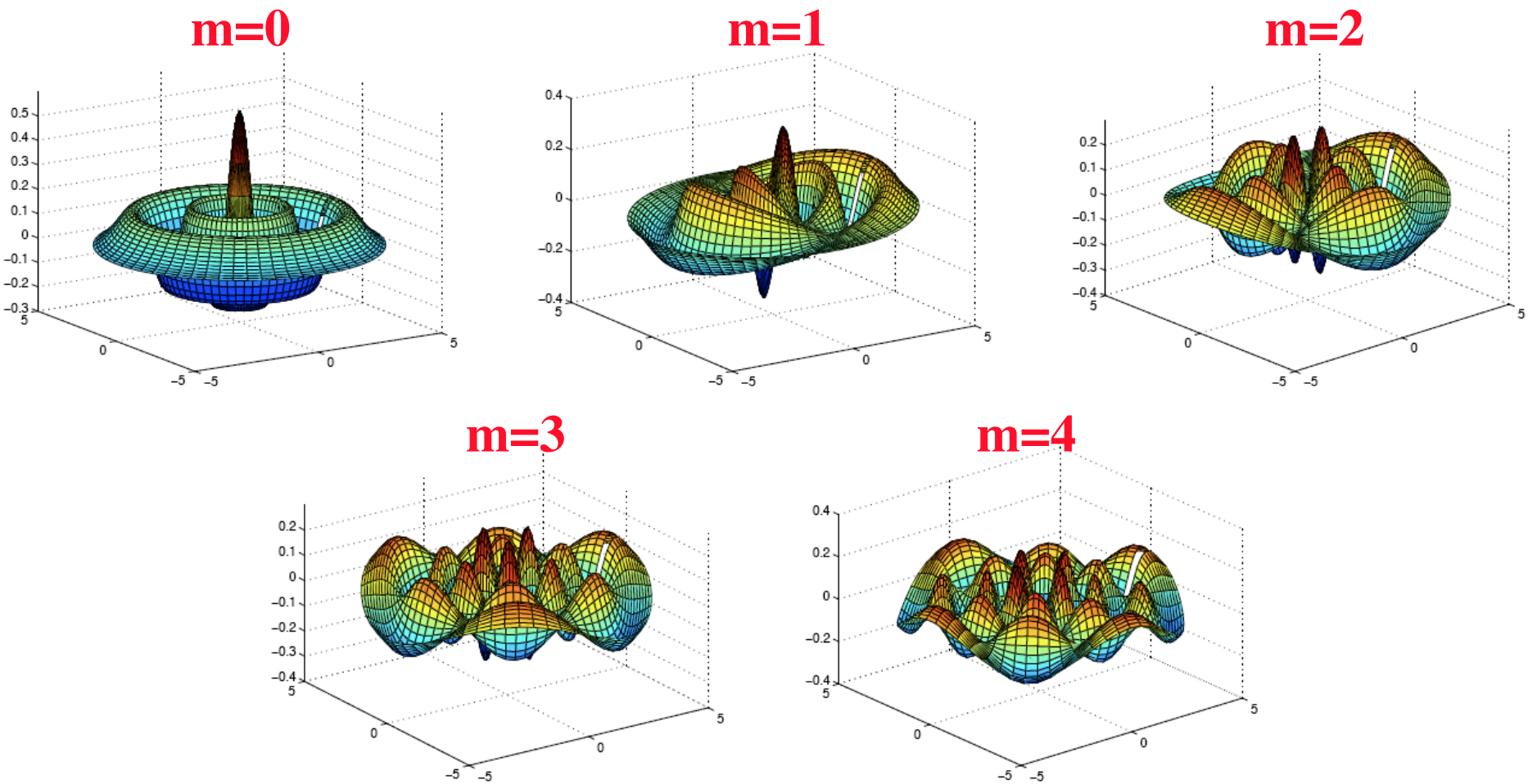
For mesons we adopt (later extended to baryons): [Y. Li, et al., PLB758 (2016)]

$$f_{\alpha=\{nml\}}\left(\vec{k}_\perp, x\right) = \phi_{nm}\left(\vec{k}_\perp / \sqrt{x(1-x)}\right) \chi_l\left(x\right)$$

ϕ_{nm} 2D-HO functions

χ_l Jacobi polynomials times $x^a (1-x)^b$

Set of Transverse 2D HO Modes for n=4

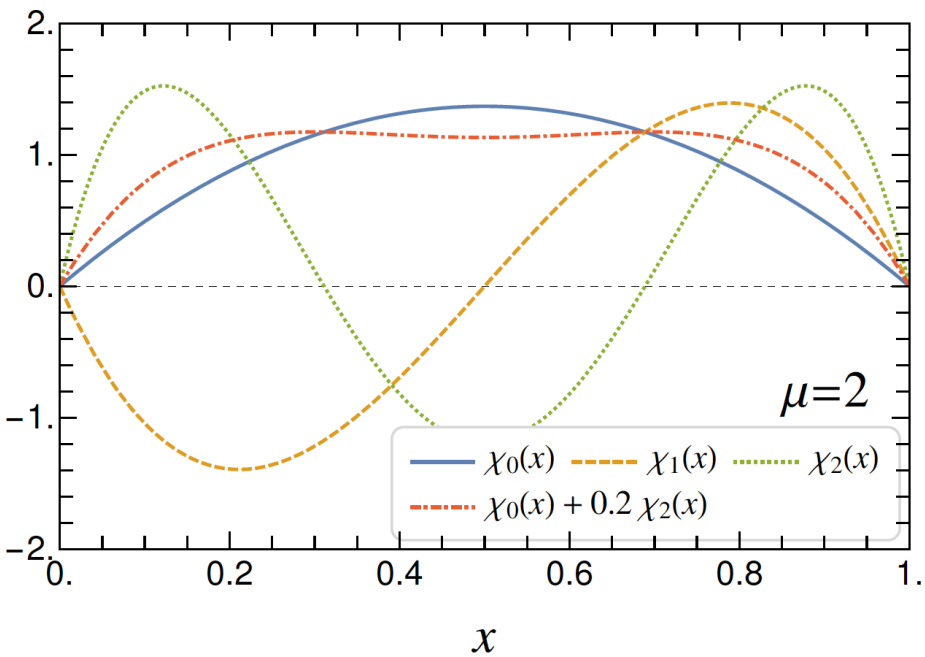


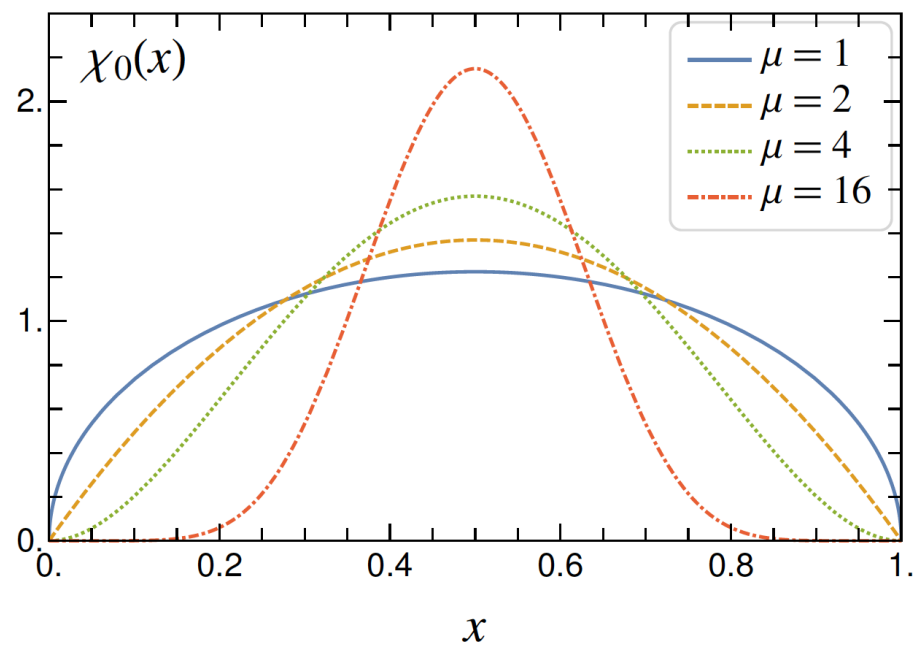
J.P. Vary, H. Honkanen, J. Li, P. Maris, S.J. Brodsky, A. Harindranath,
G.F. de Teramond, P. Sternberg, E.G. Ng and C. Yang, PRC 81, 035205 (2010)

Normalized longitudinal basis functions

$$\chi_l(x) = \frac{\sqrt{4\pi l!(2l+2\mu+1)\Gamma(l+2\mu+1)}}{\Gamma(l+\mu+1)} x^{\frac{\mu}{2}}(1-x)^{\frac{\mu}{2}} P_l^{(\mu,\mu)}(2x-1)$$

Jacobi polynomials





BLFQ
Symmetries & Constraints

Baryon number $\sum_i b_i = B$

Charge $\sum_i q_i = Q$

Angular momentum projection (M-scheme) $\sum_i (m_i + s_i) = J_z$

Longitudinal momentum (Bjorken sum rule) $\sum_i x_i = \sum_i \frac{k_i}{K} = 1$

Transverse mode regulator (2D HO) $\sum_i (2n_i + |m_i| + 1) \leq N_{\max}$

Longitudinal mode regulator (Jacobi) $\sum_i l_i \leq L$

Global Color Singlets (QCD)

Light Front Gauge

Optional Fock-Space Truncation

$H \rightarrow H + \lambda H_{CM}$

All $J \geq J_z$ states in one calculation

Finite basis regulators

Preserve transverse boost invariance

Light-Front Regularization and Renormalization Schemes

1. Regulators in BLFQ (N_{max}, L)
2. Additional Fock space truncations (if any)
3. Counterterms identified/tested*
4. Sector-dependent renormalization**
5. RGPEP (Glazek, Gomez-Rocha, and others, e.g. arXiv:1805.03436)
6. SRG & OLS in NCSM*** - adapted to BLFQ (future)

*D. Chakrabarti, A. Harindranath and J.P. Vary,
Phys. Rev. D **69**, 034502 (2004)
*P. Wiecki, Y. Li, **X. Zhao**, P. Maris and J.P. Vary,
Phys. Rev. D **91**, 105009 (2015)

****V. A. Karmanov**, J.-F. Mathiot, and A. V. Smirnov,
Phys. Rev. D **77**, 085028 (2008); Phys. Rev. D **86**, 085006 (2012)
Y. Li, **V.A. Karmanov, P. Maris and J.P. Vary,
Phys. Letts. B. 748, **278** (2015); arXiv: 1504.05233

***B.R. Barrett, P. Navratil and J.P. Vary,
Prog. Part. Nucl. Phys. **69**, 131 (2013)

Heavy Quarkonia [Y.Li,PLB758,2016; PRD96,2017]

- Effective Hamiltonian in the $q\bar{q}$ sector

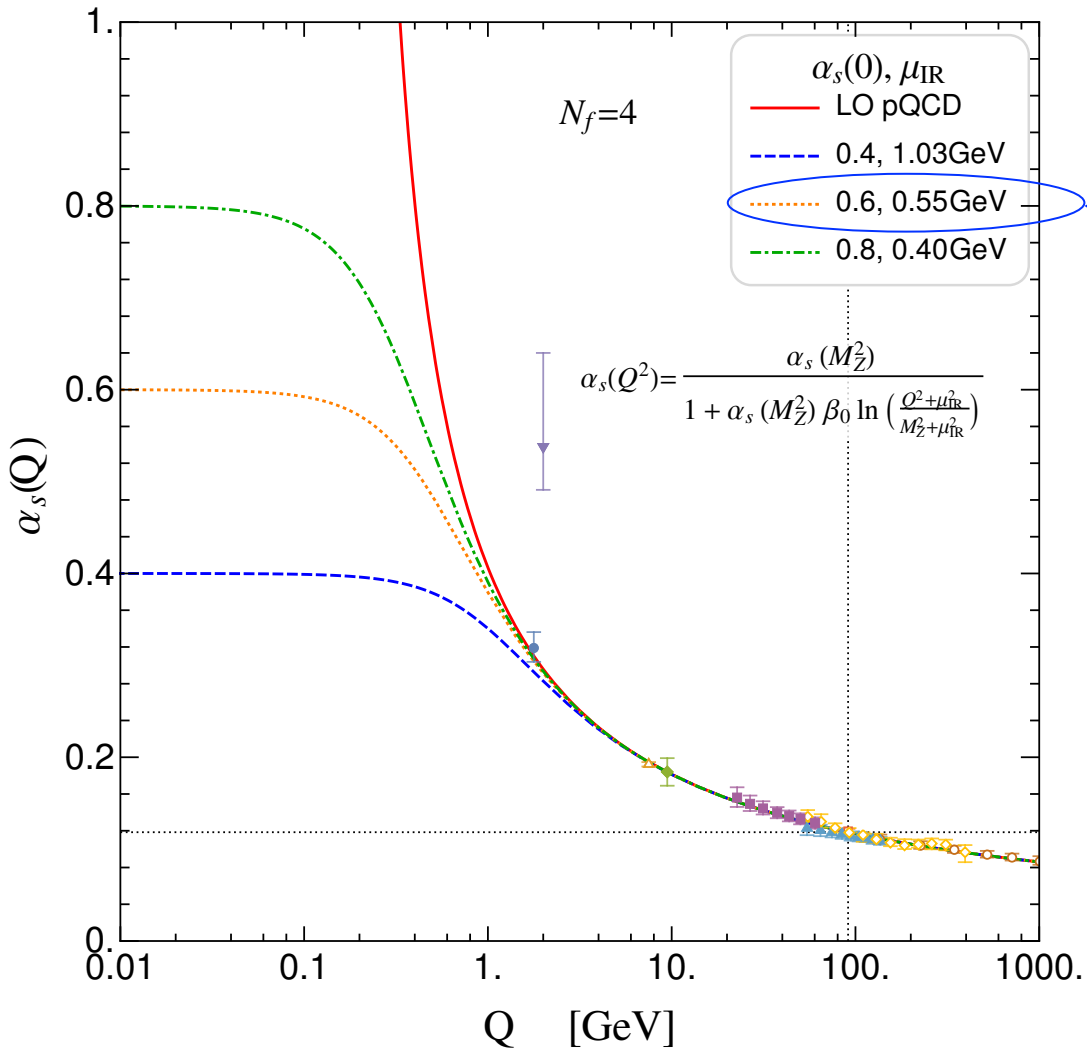
$$H_{\text{eff}} = \overbrace{\underbrace{\frac{\vec{k}_\perp^2 + m_q^2}{x} + \frac{\vec{k}_\perp^2 + m_{\bar{q}}^2}{1-x}}_{\text{LF kinetic energy}} + \underbrace{\kappa^4 x(1-x)\vec{r}_\perp^2 - \frac{\kappa^4}{(m_q+m_{\bar{q}})^2}\frac{\partial}{\partial x}\left(x(1-x)\frac{\partial}{\partial x}\right)}_{\text{confinement}}}^{H_0} + \underbrace{V_g}_{\text{one-gluon exchange}}$$

where $x = p_q^+/P^+$, $\vec{k}_\perp = \vec{p}_{q\perp} - x\vec{P}_\perp$, $\vec{r}_\perp = \vec{r}_{q\perp} - \vec{r}_{\bar{q}\perp}$.

  - Confinement
    transverse holographic confinement [S.J.Brodsky,PR584,2015]
    longitudinal confinement [Y.Li,PLB758,2016]
  - One-gluon exchange with running coupling
    $V_g = -\frac{4}{3}\frac{4\pi\alpha_s(Q^2)}{Q^2}\bar{u}_{\sigma'}\gamma^\mu u_\sigma \bar{v}_s \gamma_\mu v_{s'}$

- Basis representation
  - valence Fock sector: $|q\bar{q}\rangle$
  - basis functions: eigenfunctions of H_0 (LF kinetic energy+ confinement)

Adopt a running coupling with regulated IR behavior: improves UV properties of BLFQ applications



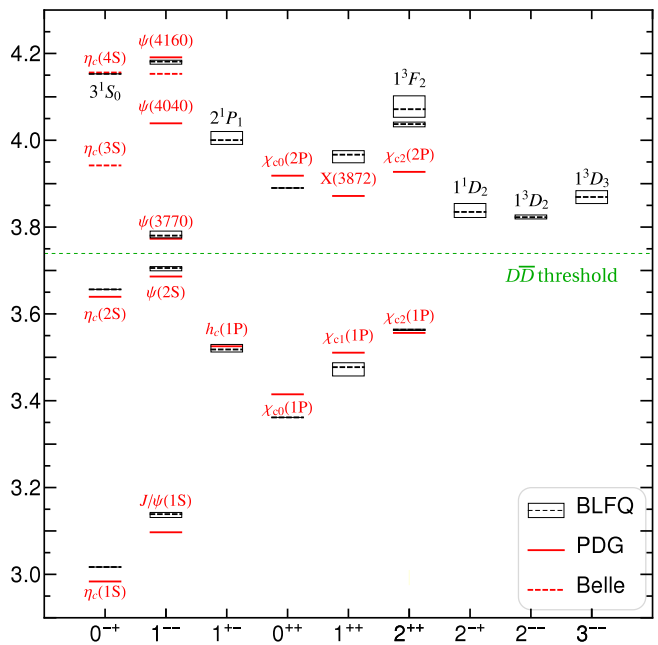
For meson results here

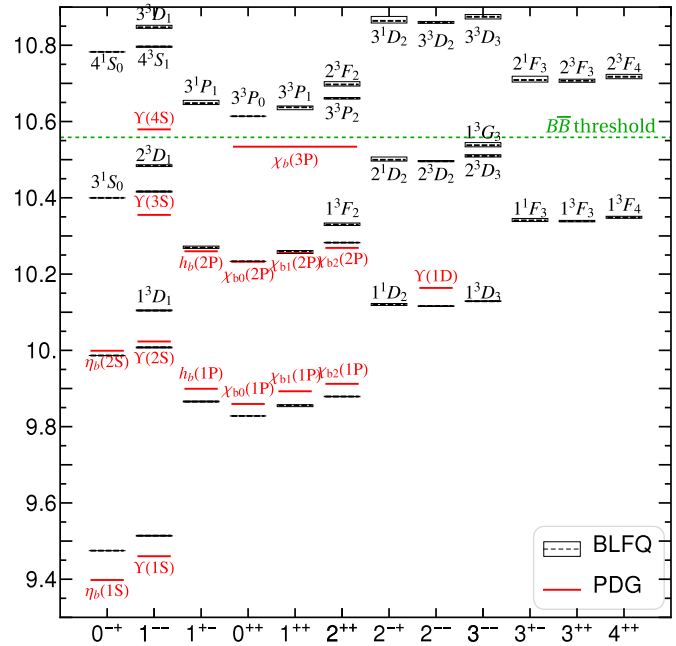
Q = average 4-momentum transfer between q and $\bar{q}$

Y. Li, P. Maris and J.P. Vary, PRD96, 016022 (2017)

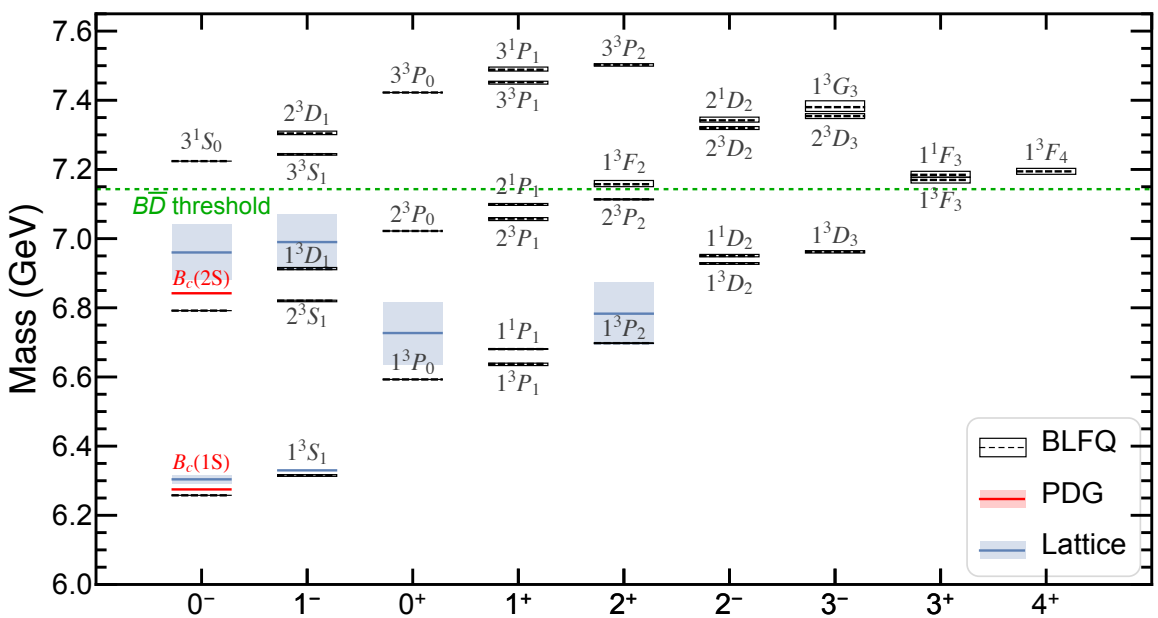
Spectroscopy

[Li, Maris & Vary, PRD '17; Tang, Li, Maris & Vary, in preparation]





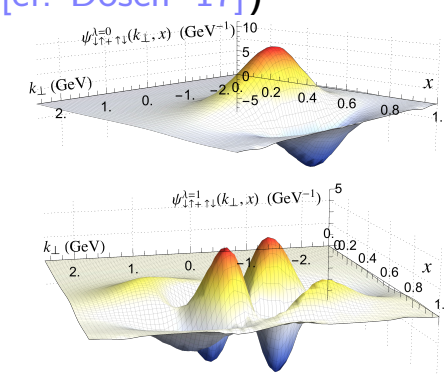
Heavy mesons:
rms deviations
31 – 38 MeV



fitting parameters:

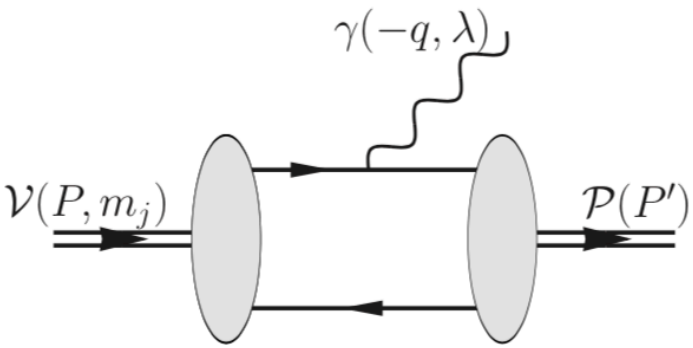
$m_c, m_b, \kappa_{qQ} = c\sqrt{M_{qQ}}$

(HQET, [cf. Dosch '17])



Radiative transitions between 0^{-+} and 1^{--} heavy quarkonia

Meijian Li, et al.; arXiv: 1803.11519

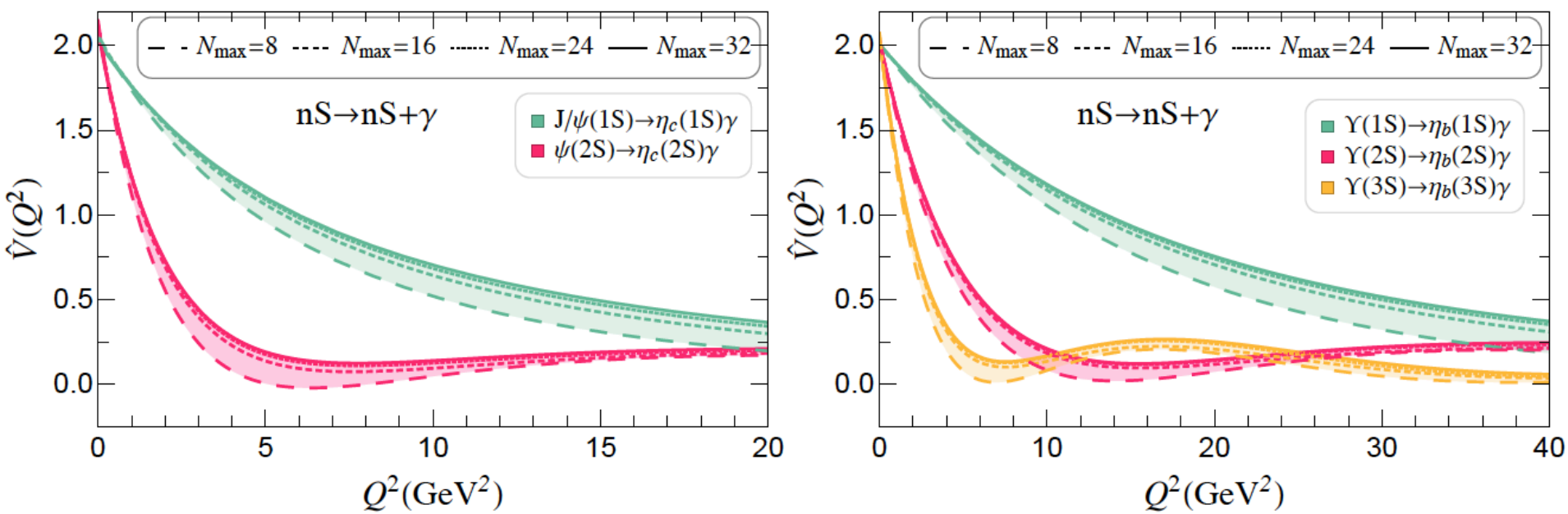


$\mathcal{V} \to \mathcal{P} + \gamma$ (or $\mathcal{P} \to \mathcal{V} + \gamma$):

$$\langle \mathcal{P}(P')|\, J^\mu(0)\, |\mathcal{V}(P,m_j)\rangle = \frac{2V(Q^2)}{m_\mathcal{P} + m_\mathcal{V}} \epsilon^{\mu\alpha\beta\sigma} P'_\alpha P_\beta e_\sigma(P,m_j)$$

momentum transfer: $q^\mu = P'^\mu - P^\mu$, $Q^2 \equiv -q^2$

Impulse approximation : $V(Q^2) = 2eQ_f \hat{V}(Q^2)$, Q_f is the quark charge.



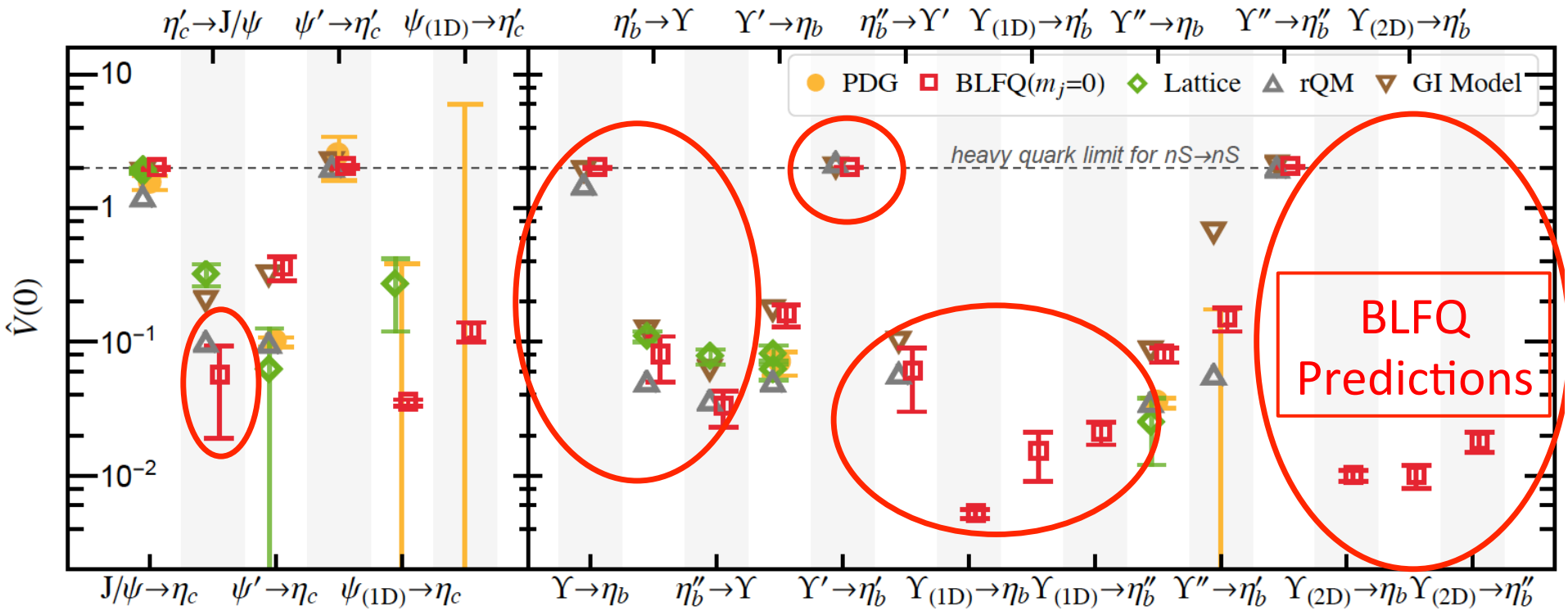
Light-front wavefunctions: Yang Li, Pieter Maris, James P. Vary. Phys. Rev. D 96, 016022 (2017)

Radiative transitions between 0^{-+} and 1^{--} heavy quarkonia

Meijian Li, et al.; arXiv: 1803.11519

Decay width:

$$\Gamma(\mathcal{V} \to \mathcal{P} + \gamma) = \int d\Omega_q \frac{1}{32\pi^2} \frac{|\vec{q}|}{m_{\mathcal{V}}^2} \frac{1}{2J_{\mathcal{V}}+1} \sum_{m_j,\lambda} |\mathcal{M}_{m_j,\lambda}|^2 = \frac{(m_{\mathcal{V}}^2 - m_{\mathcal{P}}^2)^3}{(2m_{\mathcal{V}})^3 (m_{\mathcal{P}} + m_{\mathcal{V}})^2} \frac{|V(0)|^2}{(2J_{\mathcal{V}}+1)\pi}$$



[PDG] C.Patrignani, et al., CPC40,2016.

[Lattice] J. J. Dudek, et al., PRD73,2006; PRD79, 2009. D. Bečirević, et al., JHEP01,2013; JHEP05,2015. C. Hughes, et al., PRD92,2015. R.Lewis, et al.,PRD86,2012.

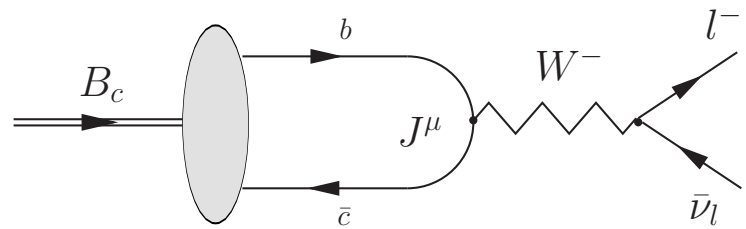
[relativistic Quark Model (rQM)] D.Ebert, et al., PRD67, 2013.

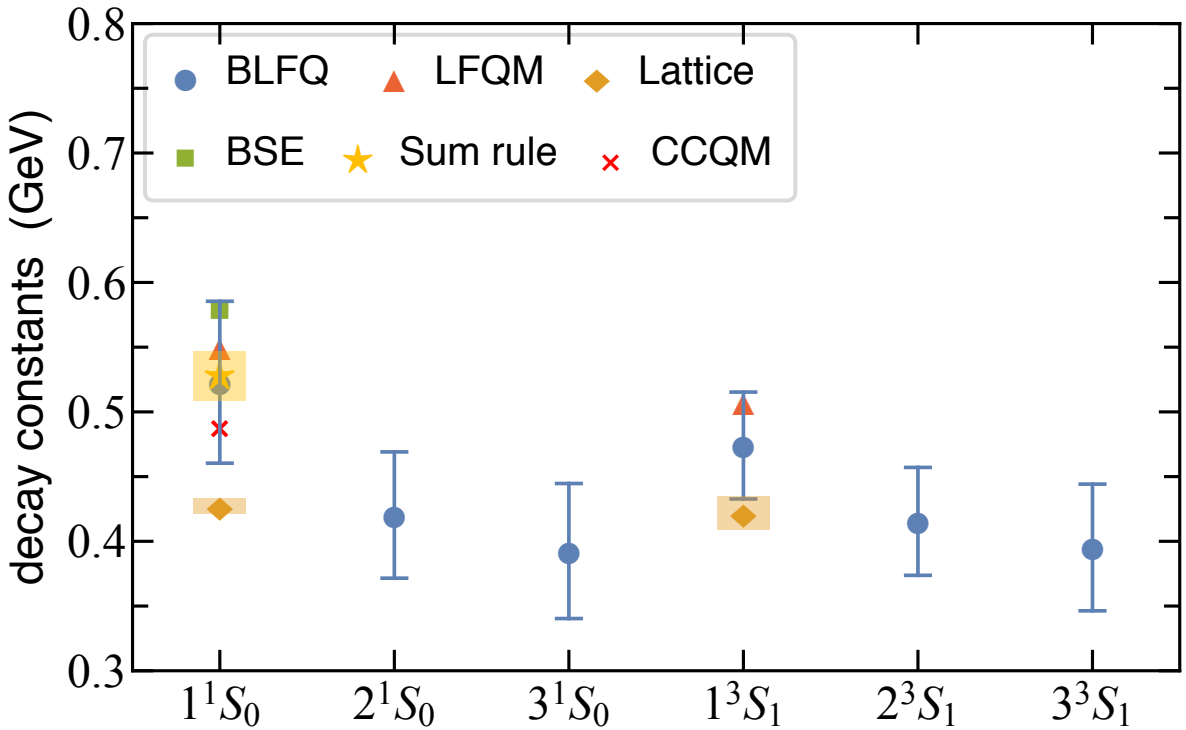
[Godfrey-Isgur Model (GI Model)] T.Barnes, et al., PRD72,2005; S.Godfrey, et al., PRD92, 2015.

B_c Meson System in BLFQ

Decay Constants

$$\frac{f_{P,V}}{2\sqrt{2N_c}} = \int_0^1 \frac{dx}{2\sqrt{x(1-x)}} \int \frac{d^2k_\perp}{(2\pi)^3} \psi^{(\lambda=0)}_{\uparrow\downarrow\mp\downarrow\uparrow}(x, \vec{k}_\perp)$$





Decay constants for vector and pseudo-scalar Bc mesons. The results are obtained with N_{max}=L_{max}=32, corresponding to UV regulators:

$$\Lambda_{\rm UV} \triangleq \kappa\sqrt{N_{\rm max}} \approx m_b + m_{\bar{c}}$$

The length of BLFQ error bars:

$$\Delta f_{b\bar{c}} = 2|f_{b\bar{c}}(N_{\rm max}=32) - f_{b\bar{c}}(N_{\rm max}=24)|$$

which is taken to indicate the sensitivity to basis truncation.

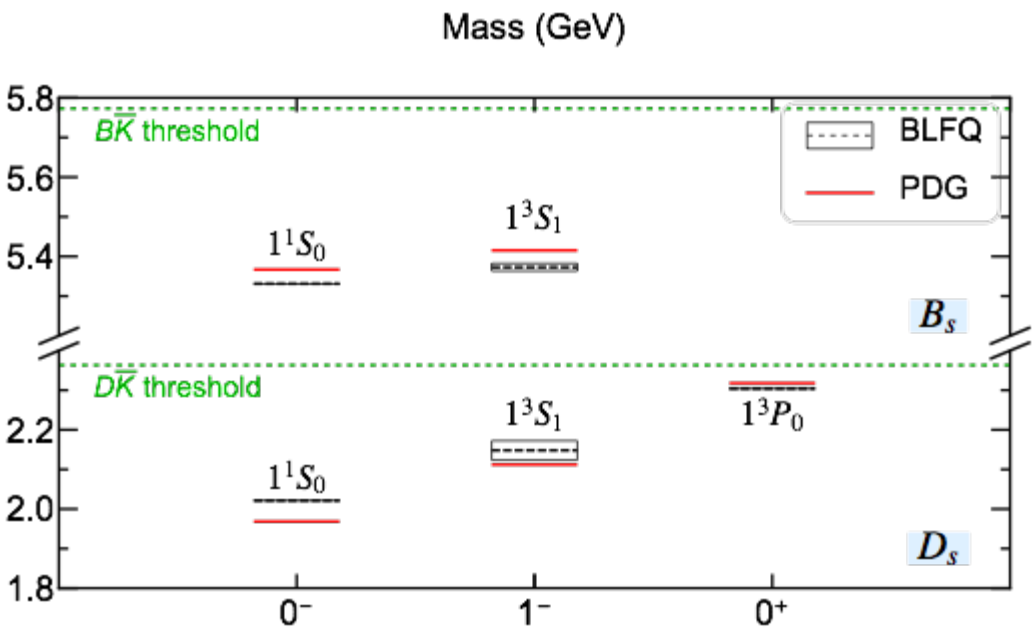
BLFQ: S. Tang, et al., in preparation
LFQM: H.-M. Cho and C.R. Ji, (2009)
Lattice: B. Colquhoun, et al., (2015)

Heavy-Light systems (preliminary)

S. Tang, et al., in preparation

	N_f	κ (GeV)	m_c (GeV)	m_b (GeV)	m_s (GeV)	rms (MeV)	$N_{\max} = L_{\max}$
$D_s(c\bar{s})$	4	0.783	1.603	–	0.597	38	32
$B_s(b\bar{s})$	4	1.054	–	4.902	0.597	39	32

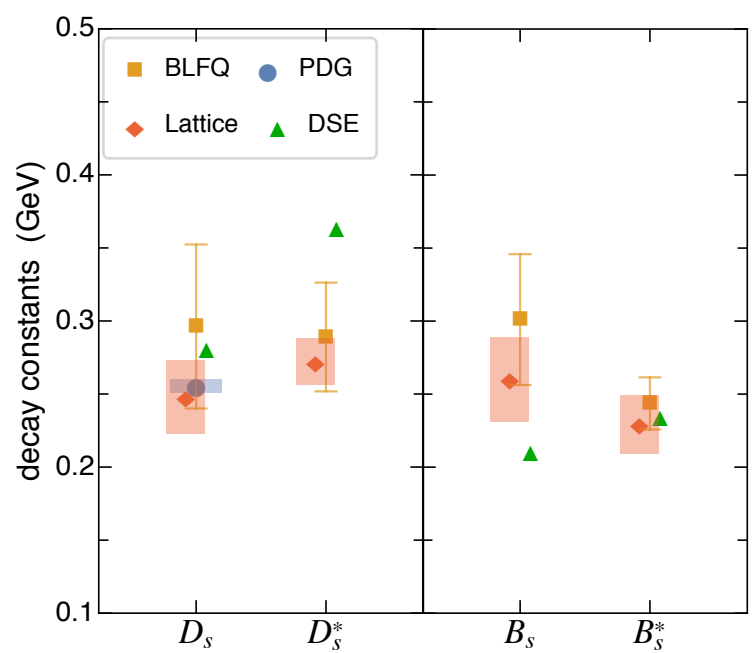
$$\kappa = \sqrt{(\kappa^2_{b\bar{b}/c\bar{c}} + \kappa^2_{s\bar{s}})/2}, \text{ with } \kappa_{s\bar{s}} = 0.54 \text{ GeV}$$



Mass spectrum of heavy-light systems. States under open flavor threshold confirmed by experiments.

Decay constants calculated with $N_{\max}$ = 8 for D_s, $N_{\max}$ = 32 for B_s, corresponding to UV cutoffs:

$$\Lambda_{\rm UV} \triangleq \kappa\sqrt{N_{\max}} \approx m_q + m_a$$



Hadron Tomography

[Adhikari, et al., Phys. Rev. C 93, 055202 (2016); & in preparation]

- Generalized parton distributions (GPDs) [Ji '97 & '98]

$$H(x,\zeta,t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \langle P'|\overline{\psi}(-\tfrac{1}{2}z)\gamma^+\psi(+\tfrac{1}{2}z)|P\rangle\Big|_{z^+=z^\perp=0}$$

$q = P' - P$, $\zeta = q^+/P^+$, $t = q^2$.

  - DVCS, SIDIS, ..., spin physics

- Impact parameter dependent GPDs: [Burkardt '01]

$$q(x,\vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} e^{i\vec{\Delta}_\perp\cdot\vec{b}_\perp} H(x,\zeta=0,t=-\Delta_\perp^2).$$

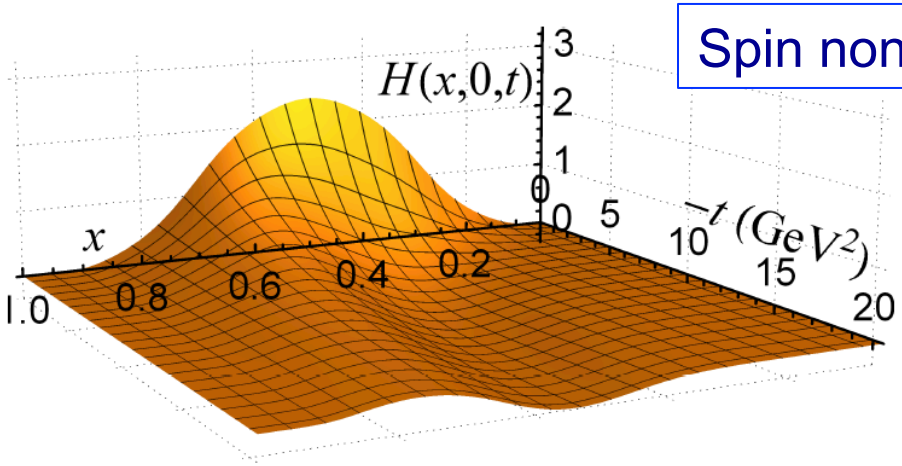
  - partonic interpretation: $\int d^2b_\perp \int_0^1 dx\, q\left(x,\vec{b}_\perp\right) = 1.$

Spin non-flip GPDs $H(x,0,t)$

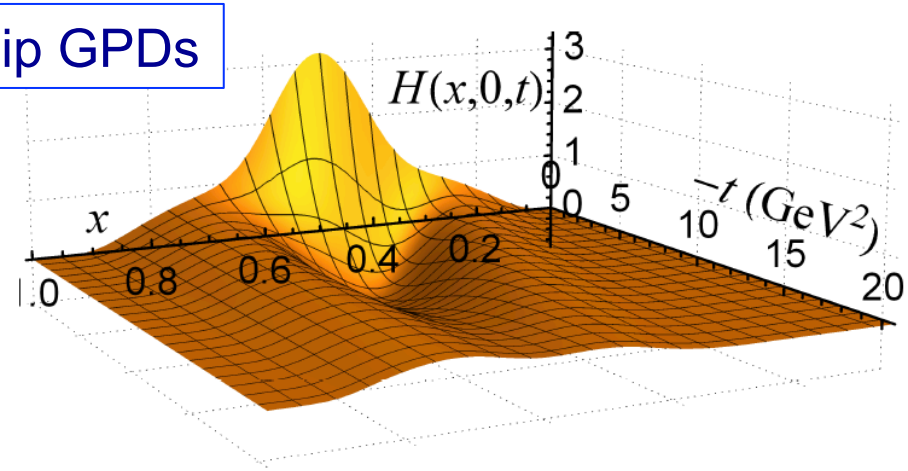
$$H(x,\xi=0,t=-\vec{\Delta}_\perp^2) = \sum_{\lambda_q,\lambda_{\bar{q}}} \int d^2\vec{k}_\perp\, \psi^*(\vec{k}'_\perp,x,\lambda_q,\lambda_{\bar{q}})\psi(\vec{k}_\perp,x,\lambda_q,\lambda_{\bar{q}}).$$

Charmonium Tomography

Spin non-flip GPDs

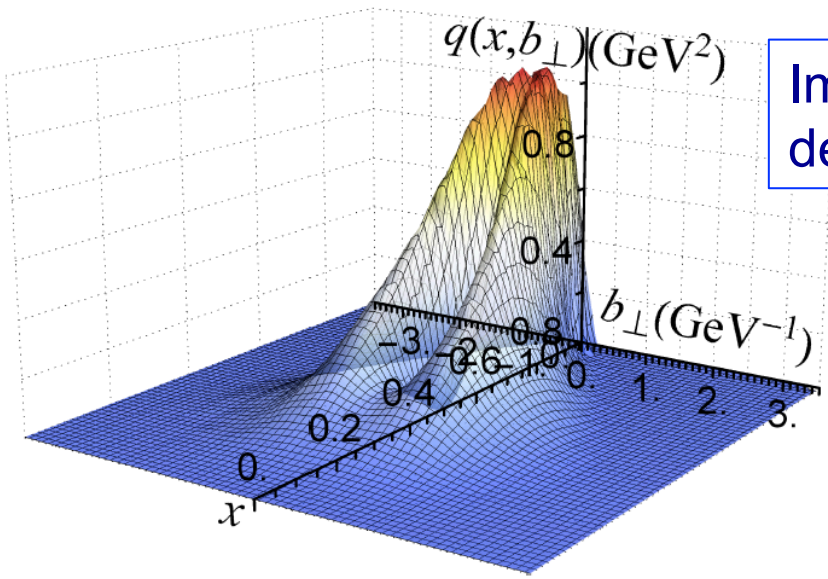


(a) $\chi_{c0} : 1^3P_0(0^{++})$

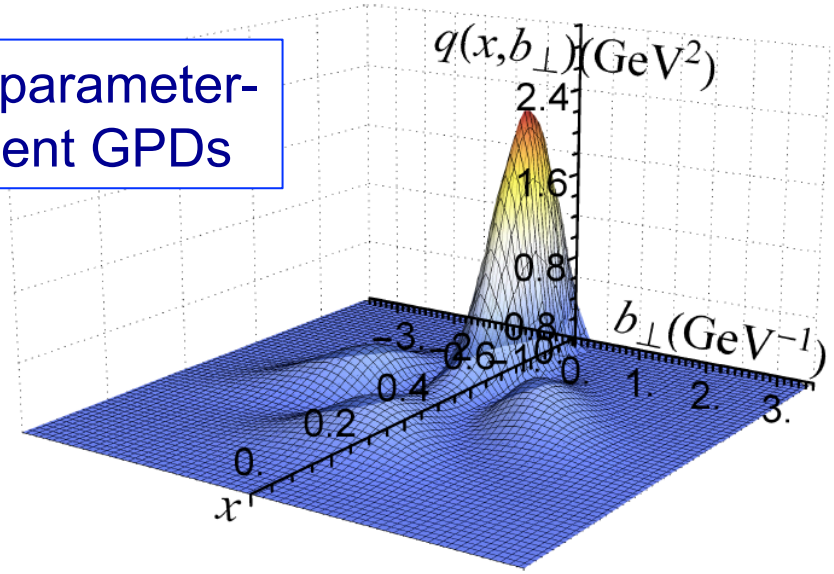


(c) $\eta_c' : 2^1S_0(0^{-+})$

Impact parameter-dependent GPDs



(b) $\chi_{c0} : 1^3P_0(0^{++})$

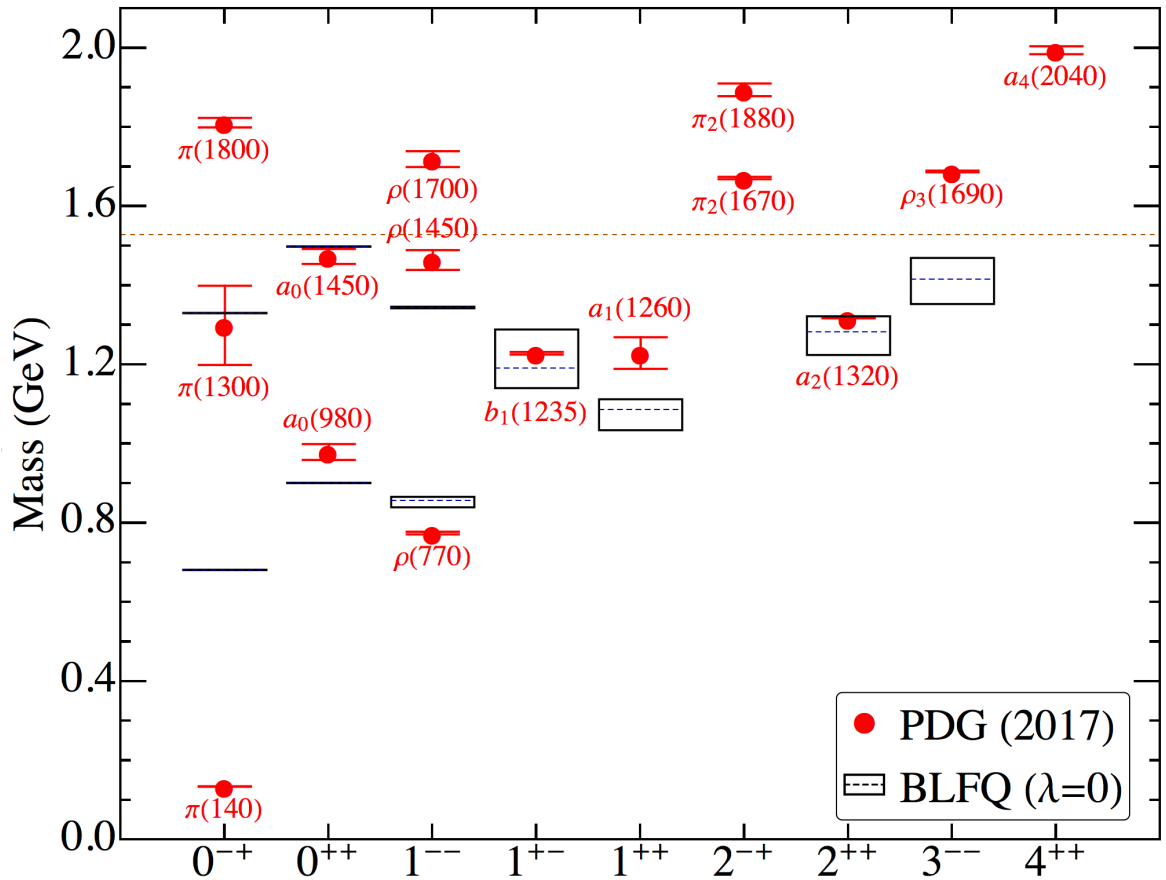


(d) $\eta_c' : 2^1S_0(0^{-+})$

L. Adhikari, et al., in preparation

Moving to light mesons – role of chiral symmetry

Spectroscopy: BLFQ with one-gluon dynamics



Confining strength and quark mass obtained by fitting the lowest PDG masses excluding pion

BLFQ mass uncertainty due to slight violation of rotational symmetry

r.m.s. deviation (9 states): 202 MeV

Model parameters:

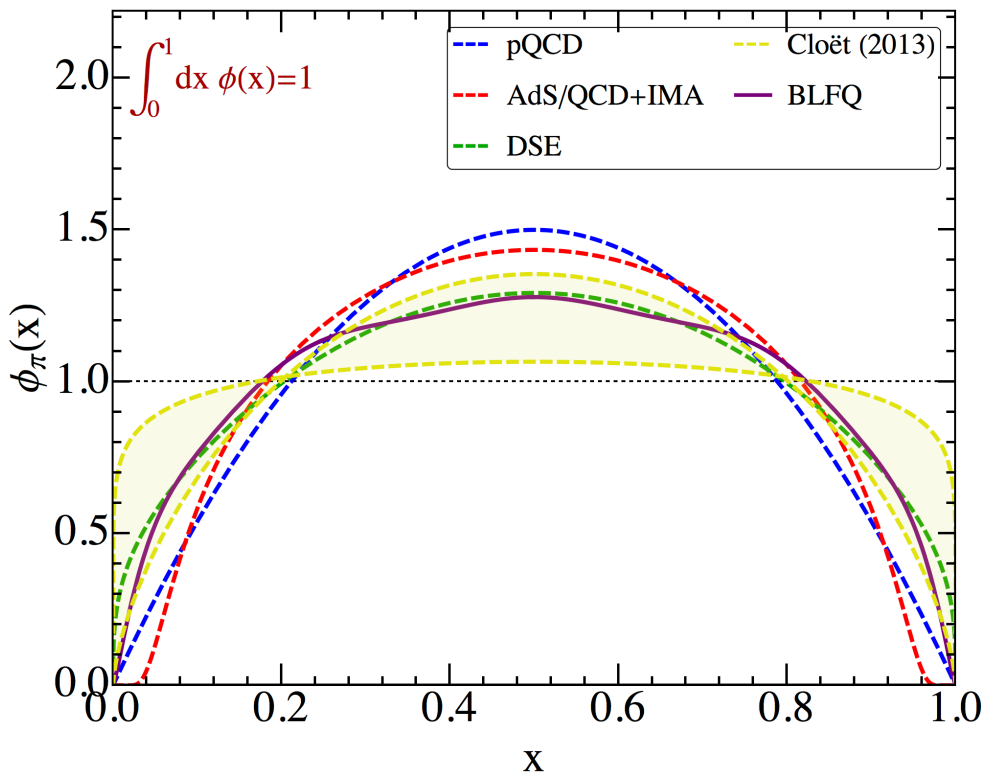
$$\kappa = 0.54 \text{ GeV}$$

$$m_q = m_{\bar{q}} = 330 \text{ MeV}$$

Wenyang Qian, et al., In preparation

Wenyang Qian, et al., In preparation

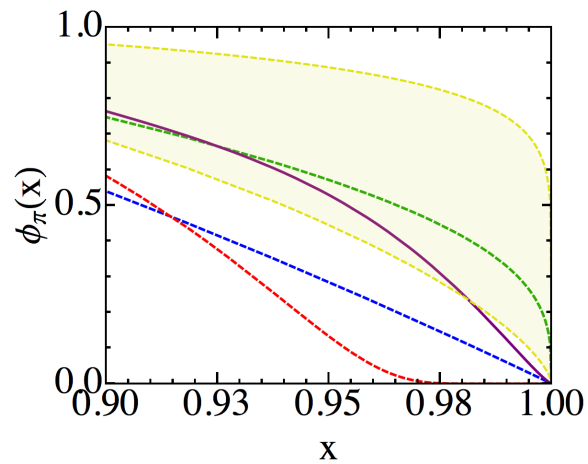
Parton distribution amplitudes for the pion



DSE: Lei Chang et al, PRL110, 132001(2013)
Cloët(2013): Cloët et al, PRL111, 092001(2013)
AdS/QCD + IMA: Brodsky et al, PhysRep548, 1(2015)

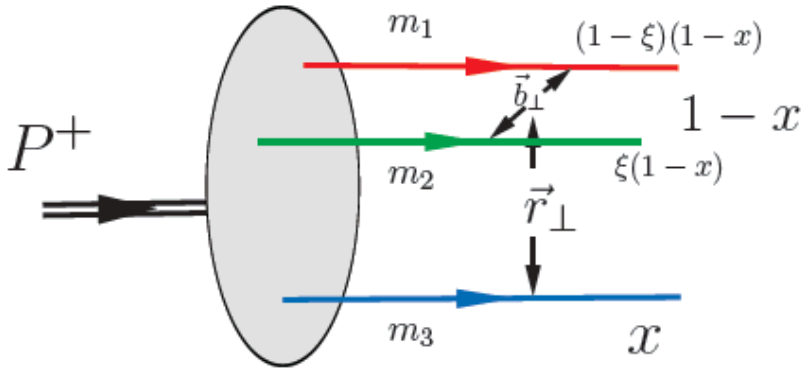
Exclusive processes at large momentum transfer

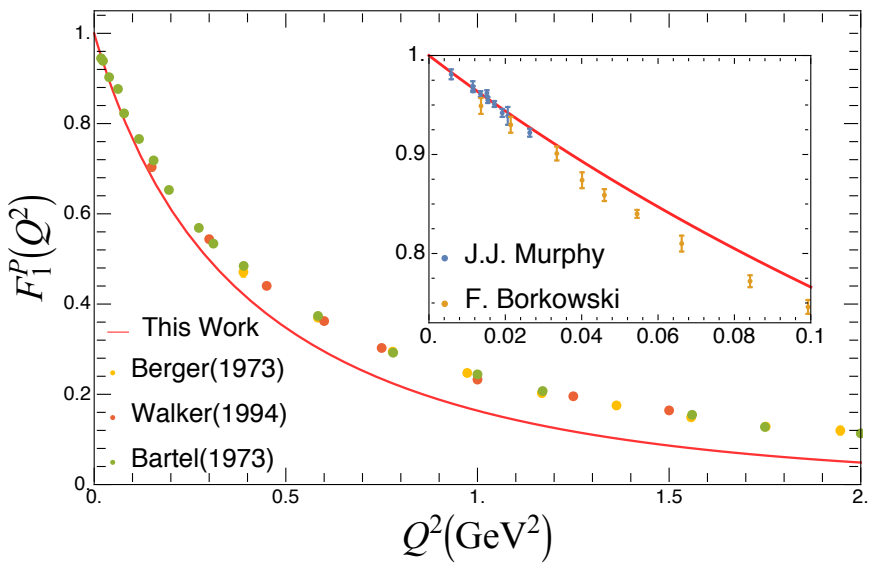
$$\phi_{\mathcal{P},\mathcal{V}}(x,\mu) \sim \frac{1}{f_{\mathcal{P},\mathcal{V}}\sqrt{x(1-x)}} \times \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^3}\psi^{(m_j=0)}_{\uparrow\downarrow\pm\downarrow\uparrow}(x,\boldsymbol{k}_\perp)$$

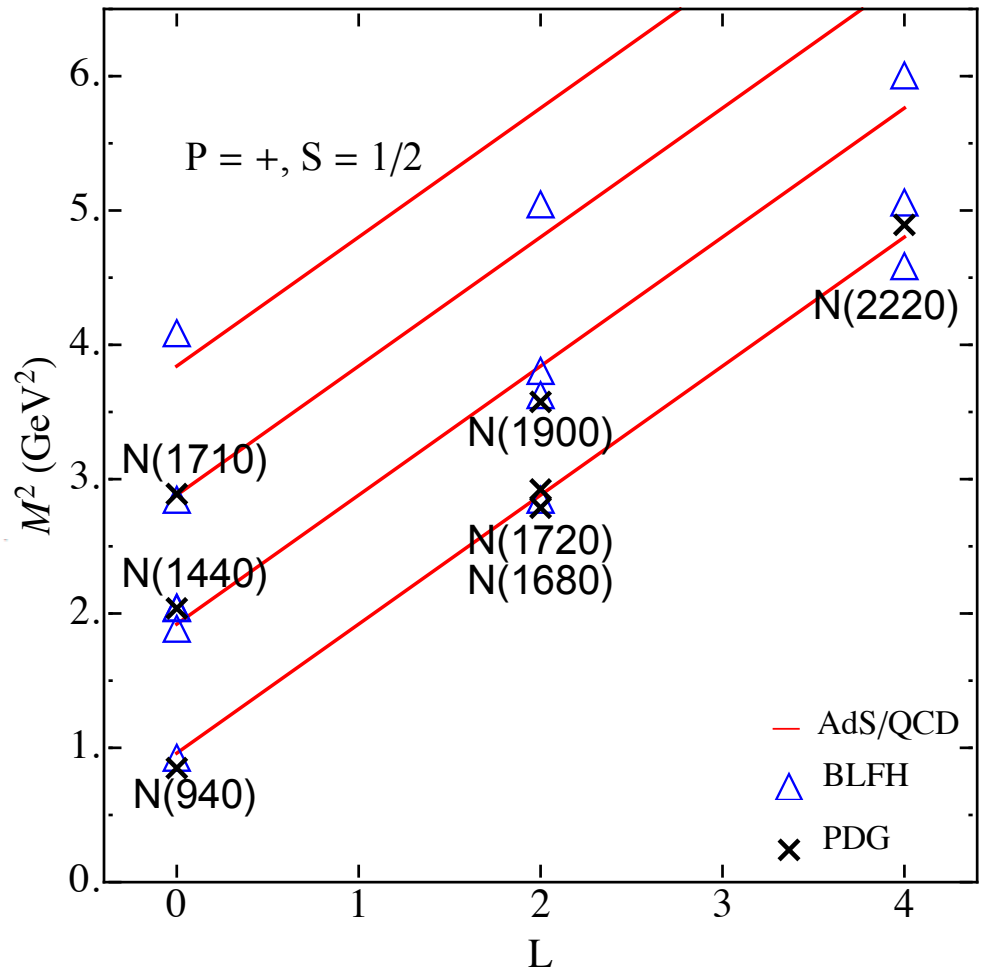


Baryons

Anji Yu, et al., in preparation







$$M^2_{\mathbf{n}_1,\mathbf{m}_1,\mathbf{n}_2,\mathbf{m}_2,L,l} = (\mathrm{m}_3 + \mathrm{M}_L)^2 + 2\kappa^2(2\mathbf{n}_1 + |\mathbf{m}_1| + 2\mathbf{n}_2 + |\mathbf{m}_2| + 2)$$
$$+ \frac{\mathrm{M}_L + \mathrm{m}_3}{\mathrm{m}_1 + \mathrm{m}_2 + \mathrm{m}_3}\kappa^2(2l+1) + \frac{\kappa^4}{(\mathrm{m}_1 + \mathrm{m}_2 + \mathrm{m}_3)^2} l(l+1) + \mathrm{const.},$$

$$\mathrm{M}_L^2 = (\mathrm{m}_1 + \mathrm{m}_2)^2 + \frac{\mathrm{m}_1 + \mathrm{m}_2}{\mathrm{m}_1 + \mathrm{m}_2 + \mathrm{m}_3}\kappa^2(2L+1) + \frac{\kappa^4}{(\mathrm{m}_1 + \mathrm{m}_2 + \mathrm{m}_3)^2} L(L+1)$$

Flavor form factor & GPD in BLFQ

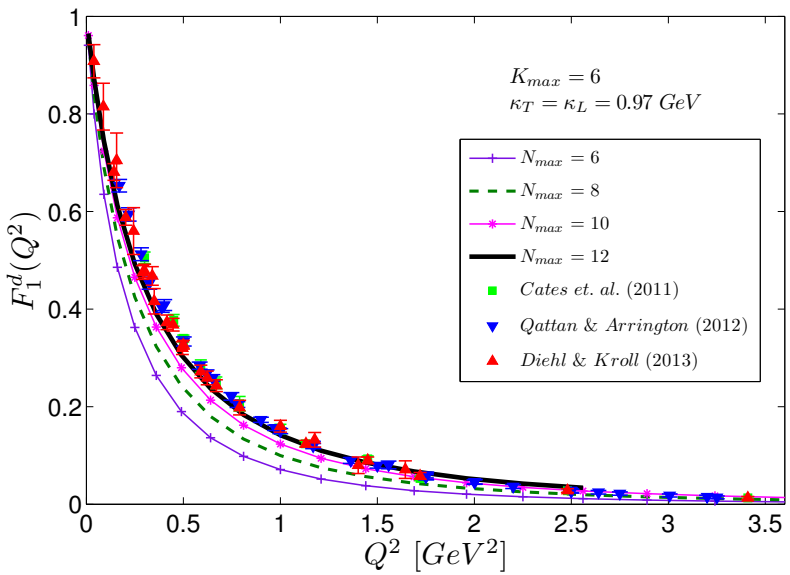
Chandan Mondal

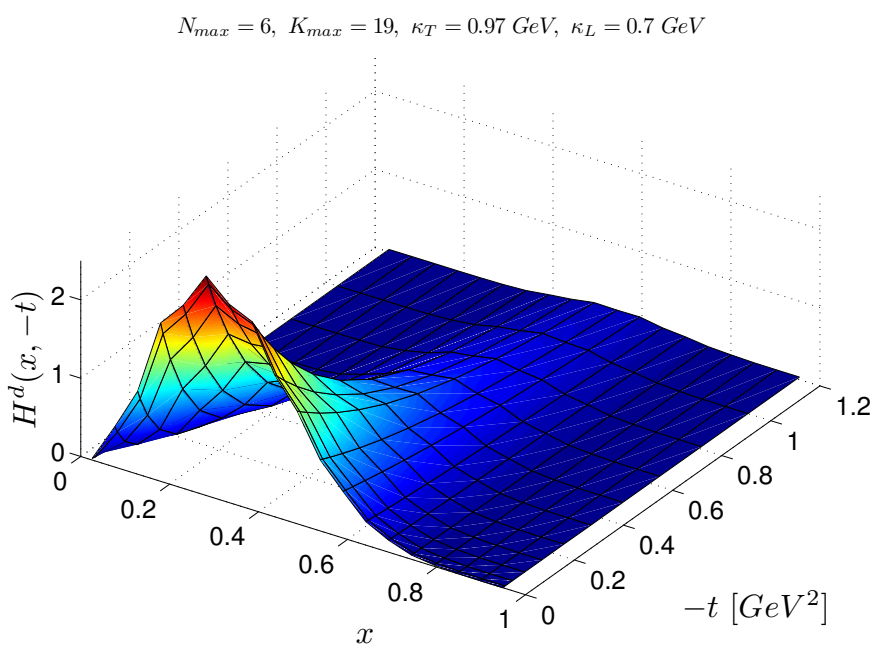
☐ Dirac form factor (F_1) in light-front [with $q^+ = 0$] for the proton

$$F_1(-q^2) = \langle P+q;\Lambda|\frac{J^+(0)}{2P^+}|P;\Lambda\rangle; \qquad F_1^q(-q^2) = \int dx\; H^q(x,-q^2).$$

✓ In terms of overlap of light-front WFs:, $H^q(x,-q^2)$:

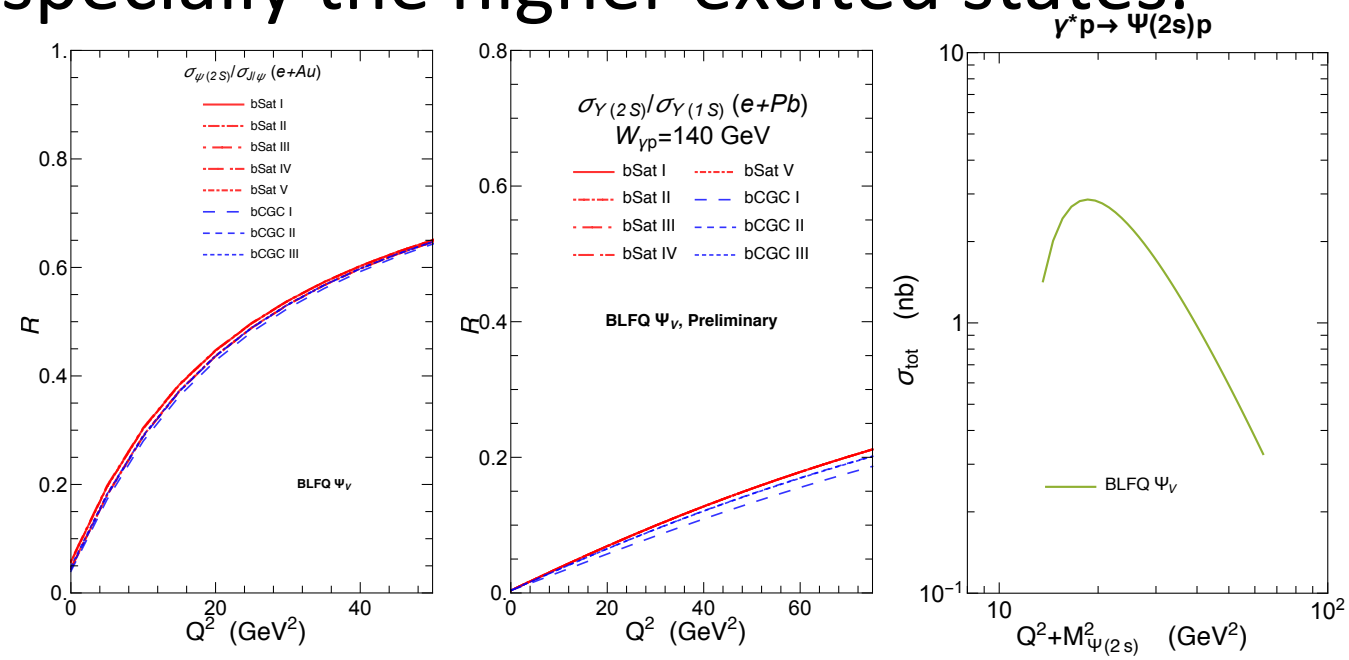
$$\sum_{\lambda_i}\int \prod_{i=1}^{3} dx_i\; d^2\mathbf{k}_{\perp i}\delta(1-\sum x_j)\delta(x-x_1)\delta^2(\sum \mathbf{k}_{\perp j})\Psi^{\Lambda *}_{\lambda_i}(x_i,\mathbf{k}'_{\perp i})\Psi^{\Lambda}_{\lambda_i}(x_i,\mathbf{k}_{\perp i})$$





Probing small-x gluons in high-energy nuclear collisions through vector-meson production

- Electron Ion Collider--high luminosity, wide kinematic range.
- Enable precision measurement of VM LFWF, especially the higher excited states.



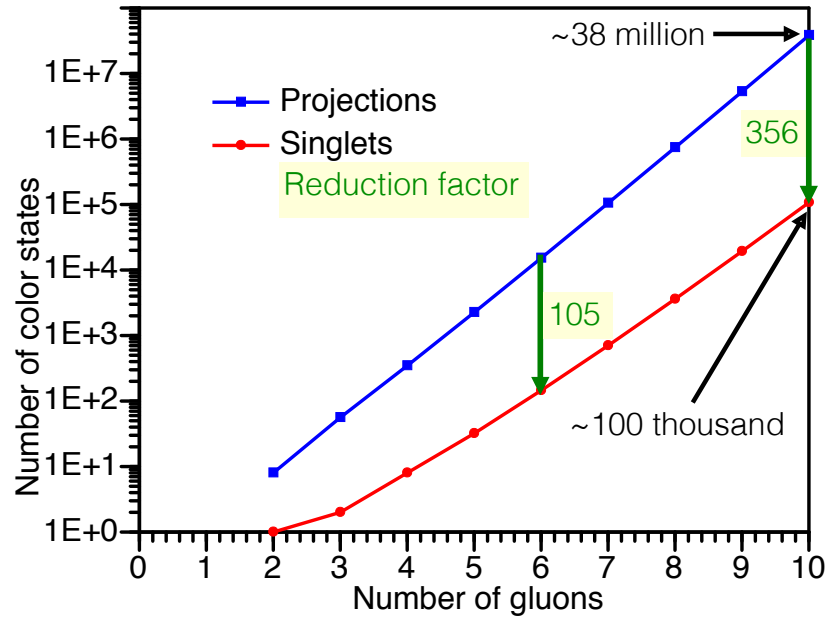
Chen, Li, Maris, Tuchin and Vary, PLB 769, 477, 2017

Full Basis Light-Front Quantization (FBLFQ)

Pure glue sector QCD – Glueballs?

Color basis space dimensions of each multi-gluon space-spin configuration

Figure extended from Vary, et al., 2010

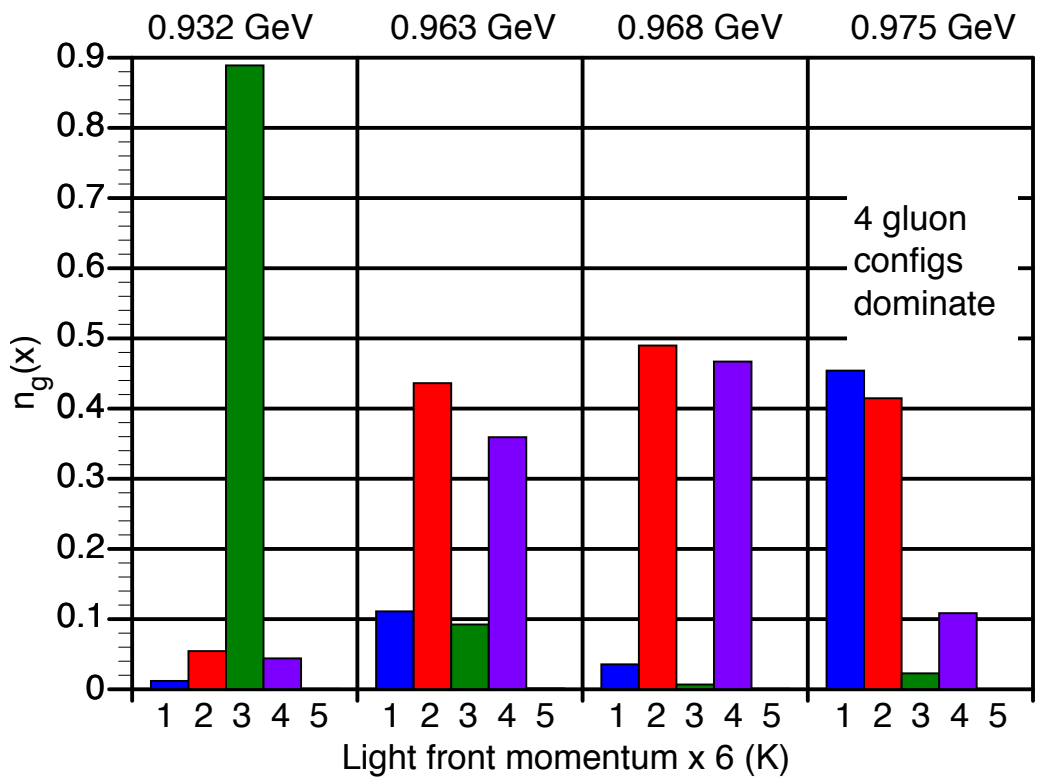


Distribution functions for 4 lowest mass eigenstates

$N_{max} = K = 6$
b=0.5 GeV
g_S=0.5
m_g = 0.25 GeV

Hamiltonian matrix dimension ~ 2000;
Calculation runs in 3 mins on laptop.

Next step: add 2 other vertices



The Lagrangian of QCD before gauge fixing:

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a, \tag{1}$$

is a result of a local $\mathrm{SU}(3)_\mathrm{c}$ gauge symmetry:

$$\psi(x) \to \exp\left(-i\sum_{a=1}^{8} T_a\Theta_a(x)\right)\psi(x).$$

Consider quark field with three flavors:

$$\psi = (\mathrm{u},\, \mathrm{d},\, \mathrm{s})^{\mathrm{T}}, \quad m = \mathrm{diag}\,\{m_\mathrm{u},\, m_\mathrm{d},\, m_\mathrm{s}\}. \tag{2}$$

In the chiral limit, there exist global $\mathrm{U}(3)_\mathrm{L} \otimes \mathrm{U}(3)_\mathrm{R}$ symmetries.

$$P_\mathrm{R} = (\mathbf{1} + \gamma_5)/2, \quad P_\mathrm{L} = (\mathbf{1} - \gamma_5)/2, \quad \psi_{\mathrm{L,R}} = P_{\mathrm{L,R}}\psi.$$

Symmetries of the strong interaction

SYMMETRY	Local gauge	Global chiral	Local chiral
Theory	QCD	NJL	Chiral EFTs
Dof	Quarks and gluons	Quarks	Mesons or baryons
Energy scale	0 to Λ_{GUT}	0 to $1\,\mathrm{GeV}$	Dof dependent

Consider color singlet four-fermion interactions in the three-flavor NJL model. The Lagrangian is given by [Klimt:1989pm]

$$\mathcal{L}^{(4)}_{\mathrm{NJL,SU(3)}} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$$
$$+ G_\pi \sum_{i=0}^{8} \left[\left(\overline{\psi}\lambda^i\psi\right)^2 + \left(\overline{\psi}i\gamma_5\lambda^i\psi\right)^2 \right]$$
$$- G_\rho \sum_{i=0}^{8} \left[\left(\overline{\psi}\gamma_\mu\lambda^i\psi\right)^2 + \left(\overline{\psi}\gamma_\mu\gamma_5\lambda^i\psi\right)^2 \right]$$
$$- G_{\mathrm{V}} \left(\overline{\psi}\gamma_\mu\psi\right)^2 - G_{\mathrm{A}} \left(\overline{\psi}\gamma_\mu\gamma_5\psi\right)^2 . \tag{3}$$

$\mathrm{SU(3)_V} \otimes \mathrm{SU(3)_A} \otimes \mathrm{U(1)_V} \otimes \mathrm{U(1)_A}$
(isospin) ~~(chiral)~~ (baryonic) ~~(axial)~~

Chiral symmetry is broken by

- nonvanishing quark mass,
- dynamics.

- The $\mathrm{U(1)_A}$ symmetry is broken by field theory effects, which can be accounted for in the NJL model by adding a determinant term:

$$\mathcal{L}_{\mathrm{det}} = G_{\mathrm{D}} \left[\det \overline{\psi}(1+\gamma_5)\psi + \det \overline{\psi}(1-\gamma_5)\psi \right] . \tag{4}$$

Determinants are taken in the flavor space, resulting in six-fermion interactions.

In the <u>two-flavor</u> scenario, the NJL Lagrangian is reduced into

$$\begin{aligned}\mathcal{L}_{\mathrm{NJL,SU(2)}} = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{G_\pi}{2}\left[(\overline{\psi}\psi)^2 - \left(\overline{\psi}\gamma_5\vec{\tau}\psi\right)^2\right] \\ - \frac{G_\rho}{2}\left[(\overline{\psi}\gamma_\mu\vec{\tau}\psi)^2 - \left(\overline{\psi}\gamma_\mu\gamma_5\vec{\tau}\psi\right)^2\right] \\ - G_{\mathrm{V}}\left(\overline{\psi}\gamma_\mu\psi\right)^2 - G_{\mathrm{A}}\left(\overline{\psi}\gamma_\mu\gamma_5\psi\right)^2, \end{aligned} \tag{5}$$

which is consistent with the three-flavor Lagrangian when determinant terms are added.

- Explicitly the two-flavor determinant terms are given by

$$\begin{aligned}&\det\,\overline{\psi}(1+\gamma_5)\psi + \det\,\overline{\psi}(1-\gamma_5)\psi \\ &= 2\left\{\bar{\mathrm{u}}\mathrm{u}\,\bar{\mathrm{d}}\mathrm{d} + \bar{\mathrm{u}}\gamma_5\mathrm{u}\bar{\mathrm{d}}\gamma_5\mathrm{d} - \bar{\mathrm{u}}\mathrm{d}\,\bar{\mathrm{d}}\mathrm{u} - \bar{\mathrm{u}}\gamma_5\mathrm{d}\,\bar{\mathrm{d}}\gamma_5\mathrm{u}\right\}.\end{aligned} \tag{6}$$

The effective Hamiltonian on the light front

The two-body interactions of the effective Hamiltonian are given by: (light front holography) $\times$ (massive fermions) $+$ (longitudinal confinement) [Li:2015zda,Li:2017mlw] :

$$H_0 = \frac{\vec{k}_\perp^2 + \mathbf{m}^2}{x} + \frac{\vec{k}_\perp^2 + \overline{\mathbf{m}}^2}{1-x} + \kappa^4 x(1-x) r_\perp^2 - \frac{\kappa^4}{(\mathbf{m}+\overline{\mathbf{m}})^2} \partial_x x(1-x)\partial_x. \tag{7}$$

We truncate the light front wavefunction at the valance Fock sector

$$\begin{aligned} &\left|\Psi_{\text{meson}}(P^+, \vec{P}^\perp, j, m_j)\right\rangle \\ &= \sum_{r,s} \int_0^1 \frac{dx}{4\pi x(1-x)} \int \frac{d\vec{\kappa}^\perp}{(2\pi)^2} \psi_{rs}(x, \vec{\kappa}^\perp) \\ &\times b_r^\dagger(xP^+, \vec{\kappa}^\perp + x\vec{P}^\perp) d_s^\dagger((1-x)P^+, -\vec{\kappa}^\perp + (1-x)\vec{P}^\perp)|0\rangle, \end{aligned} \tag{8}$$

with $P = k + p$, $x = k^+/P^+$, and $\vec{\kappa}^\perp = \vec{k}^\perp - x\vec{P}^\perp$.

The choice of basis functions

$$\psi_{rs}(x, \vec{\kappa}^{\perp}) = \sum_{nml} \psi(n, m, l, r, s) \phi_{nm} \left(\frac{\vec{\kappa}^{\perp}}{\sqrt{x(1-x)}} \right) \chi_l(x). \tag{9}$$

- The transverse basis function is given by

$$\phi_{nm}\left(\vec{q}^{\perp}; b\right) = \frac{1}{b}\sqrt{\frac{4\pi n!}{(n+|m|)!}} \left(\frac{|\vec{q}^{\perp}|}{b}\right)^{|m|} \exp\left(-\frac{\vec{q}^{\perp 2}}{2b^2}\right) L_n^{|m|}\left(\frac{\vec{q}^{\perp 2}}{b^2}\right) e^{im\varphi}, \tag{10}$$

  with $\tan(\varphi) = q^2/q^1$ and $L_n^{|m|}$ being the associated Laguerre polynomial.
- While for the longitudinal basis, we have

$$\chi_l(x; \alpha, \beta) = \sqrt{4\pi(2l+\alpha+\beta+1)} \sqrt{\frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}} \times x^{\beta/2}(1-x)^{\alpha/2} P_l^{(\alpha,\beta)}(2x-1), \tag{11}$$

  where $P_l^{(\alpha,\beta)}(z)$ is the Jacobi polynomial.

$$\alpha = \frac{2\overline{\mathbf{m}}(\mathbf{m}+\overline{\mathbf{m}})}{\kappa^2}, \qquad \beta = \frac{2\mathbf{m}(\mathbf{m}+\overline{\mathbf{m}})}{\kappa^2}. \tag{12}$$

The matrix elements for the effective Hamiltonian

$$\left\langle \Psi_{\rm meson}\left(P'^{+},\overrightarrow{P}'^{\perp},j',m'_j\right)\left|H_{\rm eff}\right|\Psi_{\rm meson}\left(P^{+},\overrightarrow{P}^{\perp},j,m_j\right)\right\rangle$$
$$= 4\pi P^{+}\delta\left(P'^{+}-P^{+}\right)(2\pi)^2\delta\left(\overrightarrow{P}'^{\perp}-\overrightarrow{P}^{\perp}\right)\sum_{r',s'}\sum_{r,s}$$
$$\times\int_0^1\frac{dx'}{4\pi x'(1-x')}\int\frac{d\overrightarrow{\kappa}'^{\perp}}{(2\pi)^2}\int_0^1\frac{dx}{4\pi x(1-x)}\int\frac{d\overrightarrow{\kappa}^{\perp}}{(2\pi)^2}$$
$$\times\,\psi^{*}_{r's'}(x',\overrightarrow{\kappa}'^{\perp})\,H_{{\rm eff}\;r's'rs}(x',\overrightarrow{\kappa}'^{\perp},x,\overrightarrow{\kappa}^{\perp})\,\psi_{rs}(x,\overrightarrow{\kappa}^{\perp}). \qquad (13)$$

When $\kappa = b$, the two-body interaction H_0 is diagonal in the basis representation:

$$\Lambda_0(n,m,l;\mathbf{m},\overline{\mathbf{m}},\kappa) = (\mathbf{m}+\overline{\mathbf{m}})^2+2\kappa^2(2n+|m|+l+3/2)+\frac{\kappa^4}{(\mathbf{m}+\overline{\mathbf{m}})^2}l(l+1). \qquad (14)$$

The matrix elements of the NJL interaction

The flavor decomposition of the direct four-fermion interaction relevant for the valence Fock sector is

$$\int dx^- \int d\overrightarrow{x}^\perp \, \overline{\psi}_{\rm Q}(x)\gamma^? \psi_{\rm Q}(x) \, \overline{\psi}_{\rm P}(x)\gamma^? \psi_{\rm P}(x)$$

$$\to \sum_{s1234} \int d\underline{k}_{1234} \, 4\pi\delta(k_1^+ + k_2^+ - k_3^+ - k_4^+)$$

$$\times (2\pi)^2 \delta\left(k_1^\perp + k_2^\perp - k_3^\perp - k_4^\perp\right)$$

$$\times \Big\{ b^\dagger_{\rm Q1} d^\dagger_{\rm Q2} d_{\rm P3} b_{\rm P4} \, \overline{u}_{\rm Q1}\gamma^? v_{\rm Q2} \, \overline{v}_{\rm P3}\gamma^? u_{\rm P4}$$

$$+ b^\dagger_{\rm P1} d^\dagger_{\rm P2} d_{\rm Q3} b_{\rm Q4} \, \overline{u}_{\rm P1}\gamma^? v_{\rm P2} \, \overline{v}_{\rm Q3}\gamma^? u_{\rm Q4}$$

$$- b^\dagger_{\rm Q1} d^\dagger_{\rm P2} d_{\rm P3} b_{\rm Q4} \overline{u}_{\rm Q1}\gamma^? u_{\rm Q4} \, \overline{v}_{\rm P3}\gamma^? v_{\rm P2}$$

$$- b^\dagger_{\rm P1} d^\dagger_{\rm Q2} d_{\rm Q3} b_{\rm P4} \overline{u}_{\rm P1}\gamma^? u_{\rm P4} \, \overline{v}_{\rm Q3}\gamma^? v_{\rm Q2} \Big\}.$$

Direct and exchange interactions are related by the Fierz transformations.

$$s \to \frac{1}{4}(s + v + \frac{t}{2} - a - p)$$

$$p \to -\frac{1}{4}(s - v + \frac{t}{2} + a - p)$$

$$v \to \frac{1}{4}(4s - 2v - 2a + 4p)$$

$$a \to -\frac{1}{4}(4s + 2v + 2a + 4p)$$

Example: scalar matrix elements for the basis expansion

$s_1' s_2' s_2 s_1$	$\langle n' m' l' s_1' s_2' \lvert \overline{u}_{\mathrm{u}} u_{\mathrm{u}}\, \overline{v}_{\mathrm{d}} v_{\mathrm{d}} \rvert n m l s_1 s_2 \rangle$
$++++$	$(-1)^{n'+n+1}(b^2/\pi)\delta_{m',0}\delta_{m,0}\mathbf{m}\overline{\mathbf{m}}\{L'(1/2,1/2)L(-1/2,-1/2)$ $+ L'(-1/2,1/2)L(1/2,-1/2) + L'(1/2,-1/2)L(-1/2,1/2)$ $+ L'(-1/2,-1/2)L(1/2,1/2)\}$
$\cdots$	$\cdots$

$$L_l(a,b;\alpha,\beta) \equiv \int_0^1 \frac{dx}{4\pi}\, x^b (1-x)^a \chi_l(x;\alpha,\beta)$$
$$= \sqrt{\frac{2l+\alpha+\beta+1}{4\pi}} \sqrt{\frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}}$$
$$\times \sum_{m=0}^{l} \binom{l+\alpha}{m} \binom{l+\beta}{l-m} (-1)^{l-m} B\left(\frac{\beta}{2}+b+m+1, \frac{\alpha}{2}+a+l-m+1\right), \tag{15}$$

where $B(s,t) = \Gamma(s)\Gamma(t)/\Gamma(s+t)$ is the Euler Beta function.

Two-flavor NJL

- Consider the interaction:

$$\mathcal{H}^{\rm eff}_{{\rm NJL}\,\pi} = -\frac{G_{{\rm P}\pi}}{2} P^+ \left[(\overline{\psi}\psi)^2 + (\overline{\psi} i\gamma_5 \vec{\tau}\,\psi)^2 \right], \tag{16}$$

  responsible for the binding of $\pi^\pm$.
- Symmetries preserved: $\mathrm{SU(2)_V} \otimes \mathrm{SU(2)_A} \otimes \mathrm{U(1)_V}$.
- The binding of $\rho^\pm$ is already taken care of by the confinement interaction in the basis diagonal Hamiltonian.

Three-flavor NJL

- For the binding of $K^\pm$:

$$\mathcal{H}^{\rm eff}_{{\rm NJL}\,K+} = -G_{{\rm P}K} P^+ [(\overline{\psi}\lambda_a\psi)^2 - (\overline{\psi}\lambda_a\gamma_5\psi)^2]. \tag{17}$$

- The expansion in flavor space is given by

$$\begin{aligned}(\overline{\psi}\,\lambda_a\gamma^?\,\psi)^2 &= (\bar{\mathrm{u}}\gamma^?\mathrm{u} + \bar{\mathrm{d}}\gamma^?\mathrm{d})^2 + (\bar{\mathrm{u}}\gamma^?\mathrm{d} + \bar{\mathrm{d}}\gamma^?\mathrm{u})^2 \\ &\quad + 2[(\bar{\mathrm{s}}\gamma^?\mathrm{s})^2 + 2\,\bar{\mathrm{u}}\gamma^?\mathrm{s}\,\bar{\mathrm{s}}\gamma^?\mathrm{u} + 2\,\bar{\mathrm{d}}\gamma^?\mathrm{s}\,\bar{\mathrm{s}}\gamma^?\mathrm{d}].\end{aligned} \tag{18}$$

With these input parameters

BLFQ-NJL	model parameters	κ	369.5 MeV	$N_{\max}$	8
$m_{\rm l}$	222.2 MeV	$G_{{\rm P}\pi}$	6.340 GeV^{-2}	$M_{\max}$	2
$m_{\rm s}$	398.8 MeV	$G_{{\rm P}K}$	7.326 GeV^{-2}	$L_{\max}$	8

,

the BLFQ-NJL model produces

	Mass (MeV)	Decay constant (MeV)	Charge radius (fm^2)
π^+	139.57	148.08	0.263
ρ^+	775.26	121.02	0.773
K^+	493.68	177.77	0.250
K^{*+}	891.76	106.30	0.608
K_0^{*+}	790.21	49.05	-

.

$\mu_{\rho^+} = \mu_{K^{*+}} = 2, \quad \mathcal{Q}_{\rho^+} = -0.0335\ \mathrm{fm}^2, \quad \mathcal{Q}_{K^{*+}} = -0.0283\ \mathrm{fm}^2$

Charge radius (fm^2)	π^+	K^+	ρ^+	K^{*+}
BLFQ-NJL	0.263	0.250	0.773	0.608
Hutauruk:2016sug	0.396	0.343		
Carrillo-Serrano:2015uca			0.67	
PDG	0.452(10)	0.256(33)		

Elastic form factors for light mesons

The matrix elements of the electromagnetic current are given by

$$\langle\psi(P', m_j')|J^\mu(x)|\psi(P, m_j)\rangle = 2P^+ \, I^\mu_{mj,mj'}(Q^2)\, e^{iq\cdot x}, \tag{19}$$

with $Q^2 = -q^2 = -(P' - P)^2$. In the Drell-Yan frame $q^+ = 0$. Using the $+$ component of the current operator, Drell-Yan-West formula:

$$I^+_{mj,mj'}(Q^2) = \sum_{ss'} \int_0^1 \frac{dx}{4\pi x(1-x)} \int \frac{d\vec{k}^\perp}{(2\pi)^2} \times \psi^*_{mj'}(k^\perp + (1-x)q^\perp, x, s, s')\, \psi_{mj}(k^\perp, x, s, s'), \tag{20}$$

where $Q^2 = -q^{\perp 2}$.

- For pseudoscalar mesons, $G_0(Q^2) = I^+_{0,0}(Q^2)$.
- For vector mesons, Grach and Kondratyuk prescription:

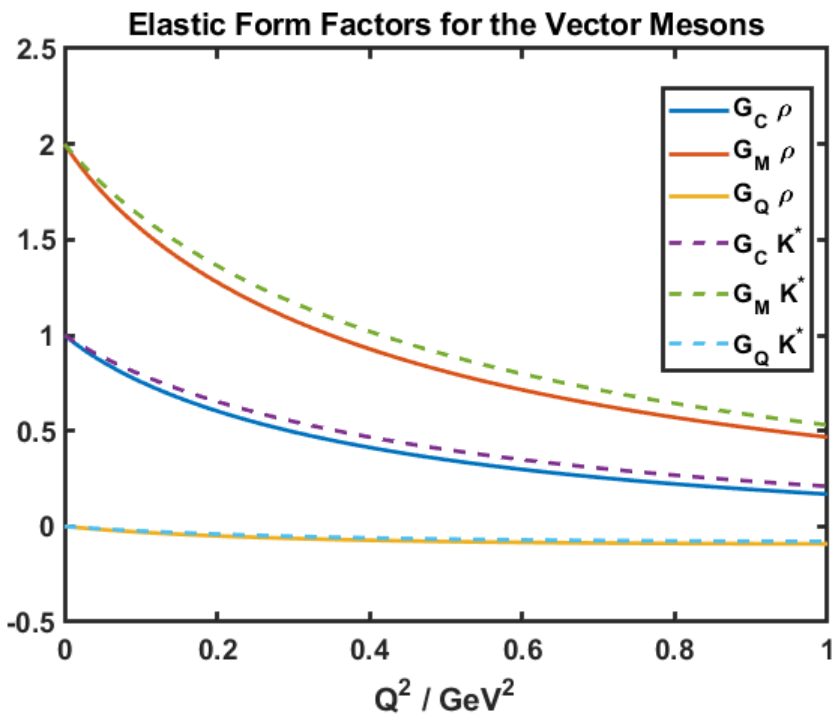
$$G_{\mathrm{E}} = \frac{1}{3}\left[(3-2\eta) I^+_{1,1} + 2\sqrt{2\eta}\, I^+_{1,0} + I^+_{1,-1}\right] \tag{21}$$

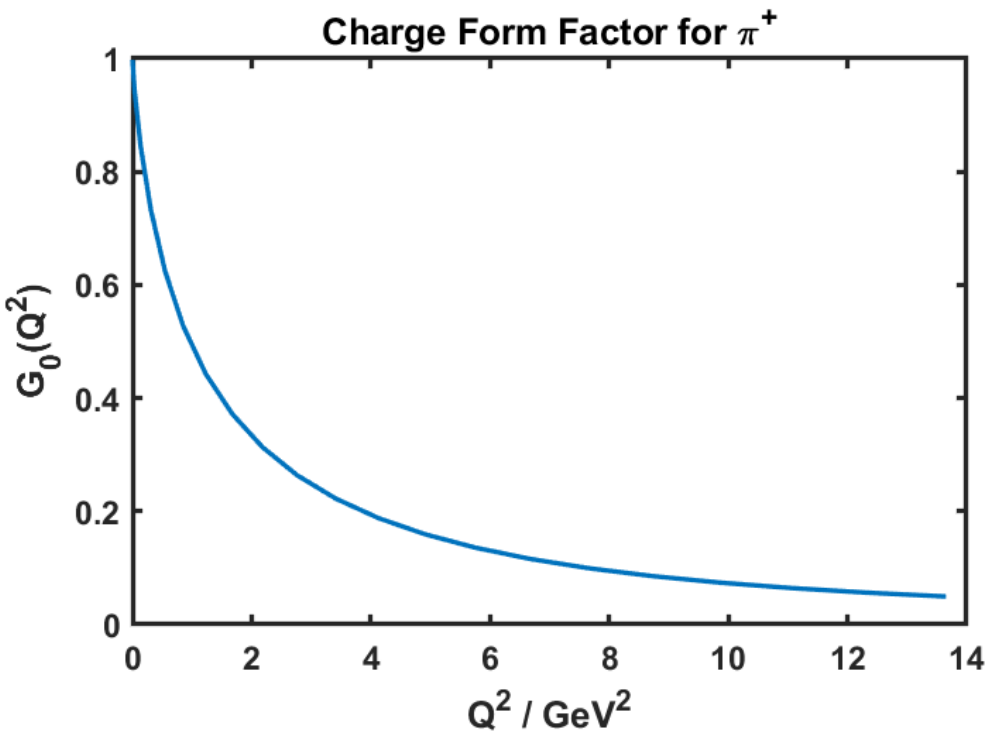
$$G_{\mathrm{M}} = 2\left[I^+_{1,1} - I^+_{1,0}/\sqrt{2\eta}\right] \tag{22}$$

$$G_{\mathrm{Q}} = \frac{2\sqrt{2}}{3}\left[-\eta\, I^+_{1,1} + \sqrt{2\eta}\, I^+_{1,0} - I^+_{1,-1}\right], \tag{23}$$

with $\eta = Q^2/(4m_{\mathrm{h}}^2)$.

Elastic form factors for light mesons

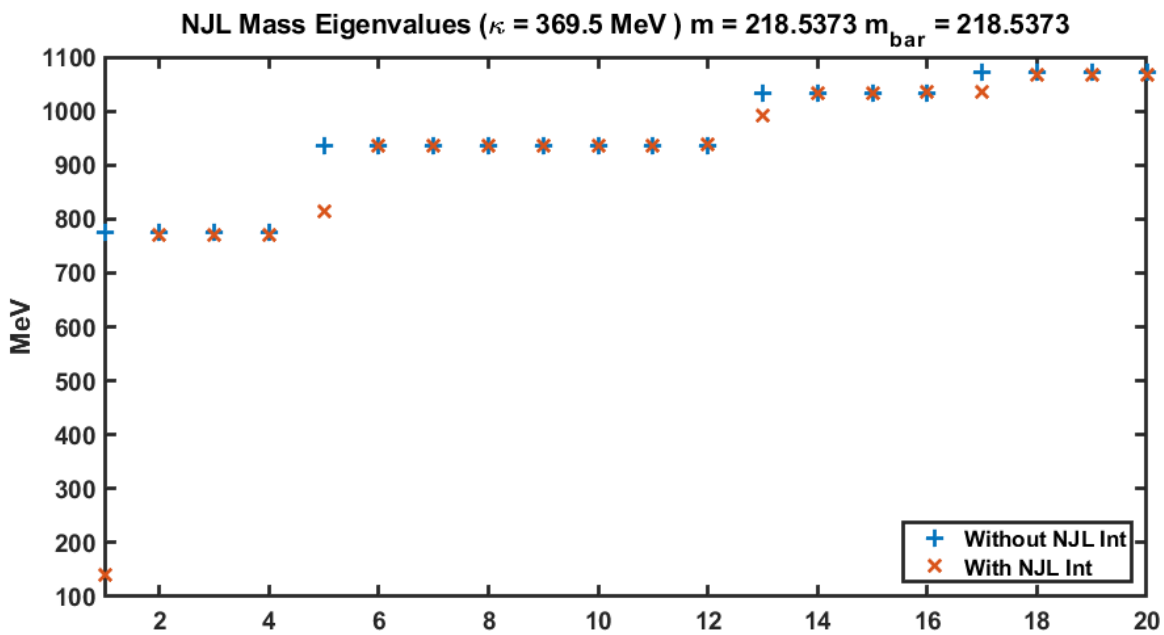


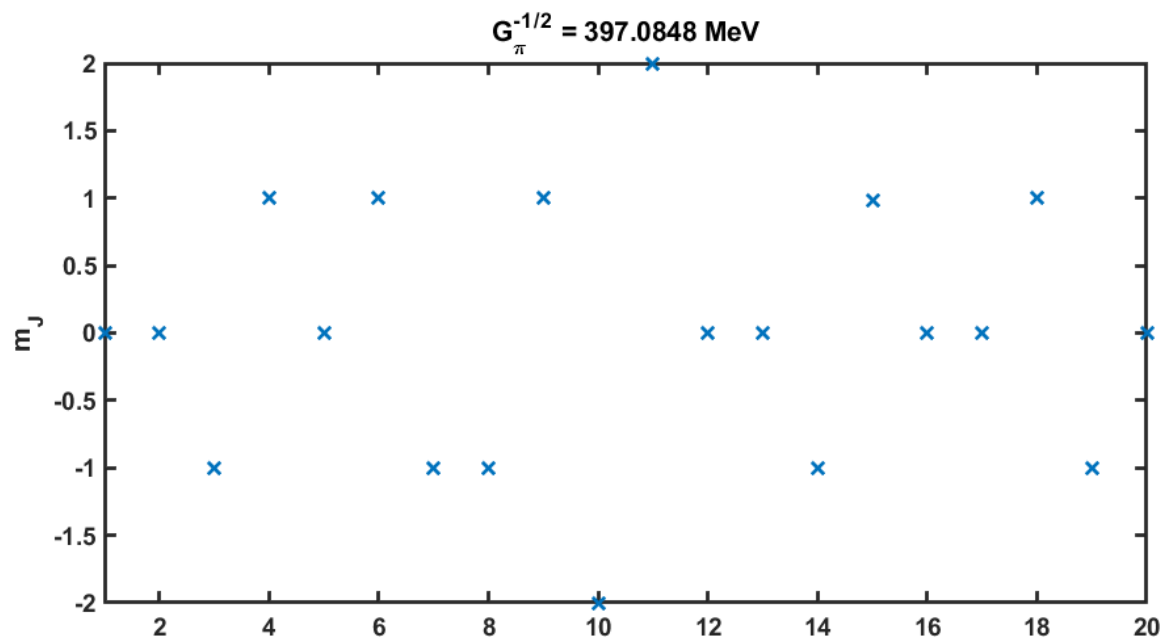


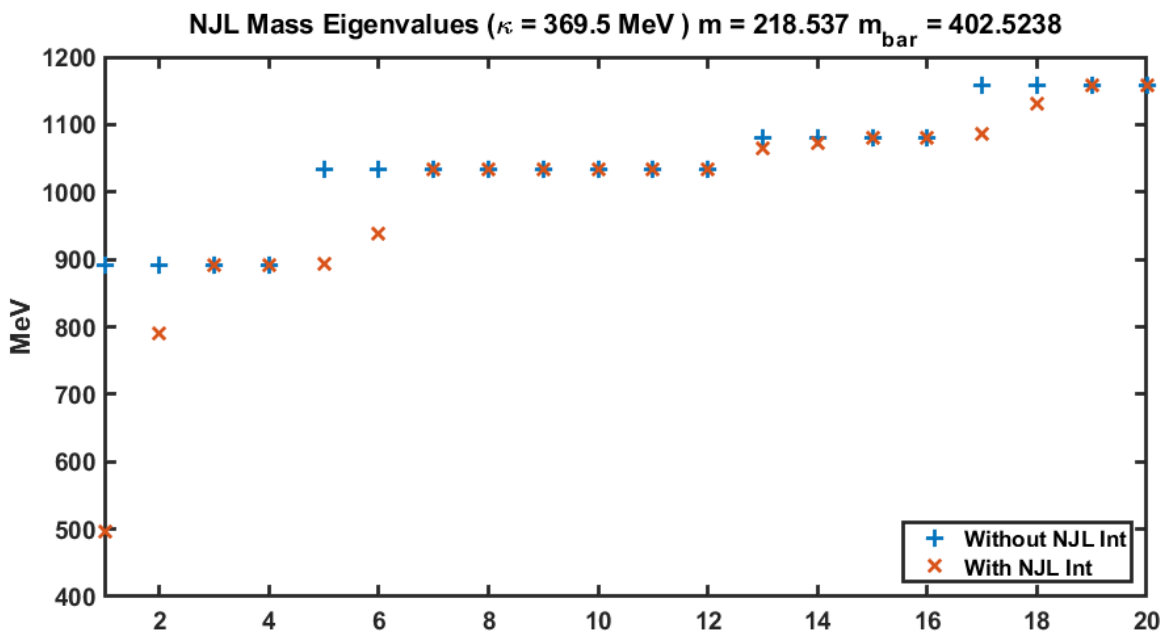
For vector mesons, the wavefunctions are (almost entirely) given by

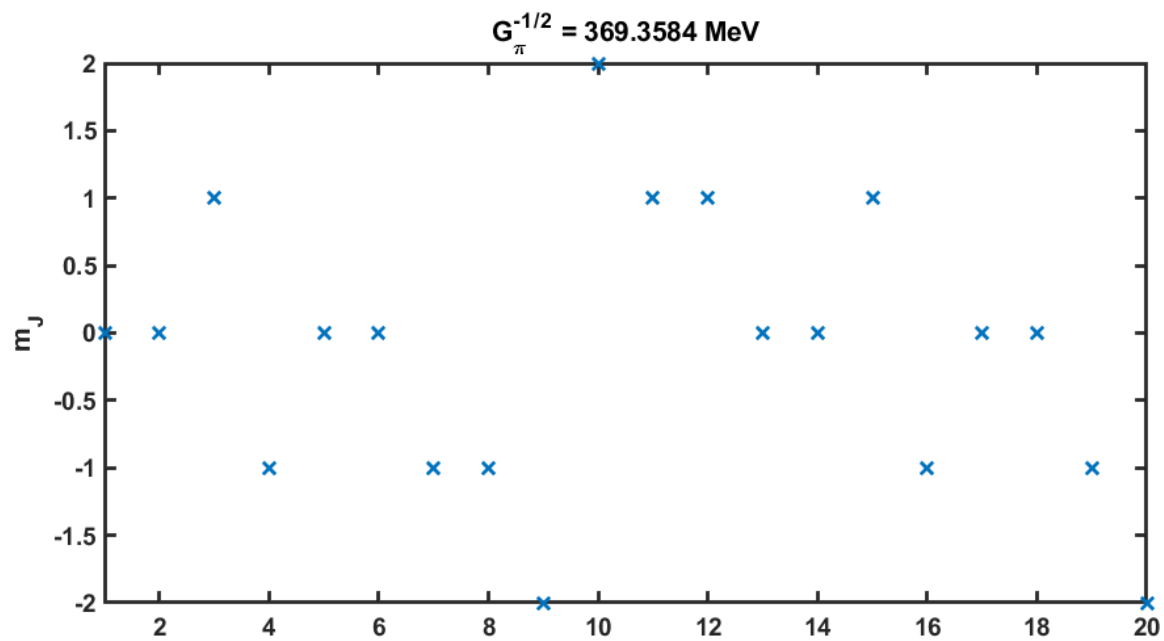
$$\psi(k^{\perp}, x) \sim x^{\beta/2}(1-x)^{\alpha/2} \exp\left(-\frac{k^{\perp 2}}{2b^2}\right) \times \begin{cases} |++\rangle \\ |+-\rangle + |-+\rangle \\ |--\rangle \end{cases} . \quad (24)$$

- Flavor singlet mesons $\eta,\ \eta',\ \omega$ and ϕ
- Neutral Kaons $K^0,\ \overline{K}^0$
- Self-energy corrections
- Excited states
- Higher Fock sector contributions, sea quarks and gluons


NJL Mass Eigenvalues (κ = 369.5 MeV) m = 218.5373 m_{bar} = 218.5373
MeV
1100
1000
900
800
700
600
500
400
300
200
100
2
4
6
8
10
12
14
16
18
20
Without NJL Int
With NJL Int


$G_{\pi}^{-1/2}$ = 397.0848 MeV
m_J
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
2
4
6
8
10
12
14
16
18
20


NJL Mass Eigenvalues (κ = 369.5 MeV) m = 218.537 m_{bar} = 402.5238
MeV
1200
1100
1000
900
800
700
600
500
400
2
4
6
8
10
12
14
16
18
20
Without NJL Int
With NJL Int


$G_{\pi}^{-1/2}$ = 369.3584 MeV
m_J
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
2
4
6
8
10
12
14
16
18
20

Spin matrix elements for the direct scalar interaction

$s_1' s_2' s_2 s_1$	$\overline{u}_{\mathrm{u}s1'}(p_1')u_{\mathrm{u}s1}(p_1)\,\overline{v}_{\mathrm{d}s2}(p_2)v_{\mathrm{d}s2'}(p_2')$
$++++$	$-\mathbf{m}\overline{\mathbf{m}}\left(\sqrt{\frac{x'}{x}}+\sqrt{\frac{x}{x'}}\right)\left(\sqrt{\frac{1-x'}{1-x}}+\sqrt{\frac{1-x}{1-x'}}\right)$
$+++-$	$\overline{\mathbf{m}}\left(\sqrt{\frac{x'}{1-x'}}q^{\mathrm{L}}-\sqrt{\frac{x}{1-x}}q'^{\mathrm{L}}\right)(2-x'-x)$
$++-+$	$\mathbf{m}(x'+x)\left(\sqrt{\frac{1-x}{x}}q'^{\mathrm{L}}-\sqrt{\frac{1-x'}{x'}}q^{\mathrm{L}}\right)$
$++--$	$-(x'+x-2x'x)q'^{\mathrm{L}}q^{\mathrm{L}}$ $+\sqrt{x'(1-x')x(1-x)}(q'^{\mathrm{L}2}+q^{\mathrm{L}2})$
$+-++$	$\mathbf{m}(x'+x)\left(\sqrt{\frac{1-x'}{x'}}q^{\mathrm{R}}-\sqrt{\frac{1-x}{x}}q'^{\mathrm{R}}\right)$
$+-+-$	$(1-x')xq'^{\mathrm{L}}q^{\mathrm{R}}+x'(1-x)q'^{\mathrm{R}}q^{\mathrm{L}}$ $-\sqrt{x'(1-x')x(1-x)}(q'^{\mathrm{L}}q'^{\mathrm{R}}+q^{\mathrm{L}}q^{\mathrm{R}})$

$s_1' s_2' s_2 s_1$	$\overline{u}_{\mathrm{u}s1'}(p_1')u_{\mathrm{u}s1}(p_1)\,\overline{v}_{\mathrm{d}s2}(p_2)v_{\mathrm{d}s2'}(p_2')$
$+-\,-+$	$++\,++$
$+-\,--$	$+++-$
$-+\,++$	$-\overline{\mathrm{m}}\left(\sqrt{\frac{x'}{1-x'}}q^{\mathrm{R}}-\sqrt{\frac{x}{1-x}}q'^{\mathrm{R}}\right)(2-x'-x)$
$-++-$	$+++\,+$
$-+\,-+$	$x'(1-x)q'^{\mathrm{L}}q^{\mathrm{R}}+x(1-x')q'^{\mathrm{R}}q^{\mathrm{L}}$ $-\sqrt{x'(1-x')x(1-x)}(q'^{\mathrm{L}}q'^{\mathrm{R}}+q^{\mathrm{L}}q^{\mathrm{R}})$
$-+\,--$	$++\,-+$
$--\,++$	$-(x'+x-2x'x)q'^{\mathrm{R}}q^{\mathrm{R}}$ $+\sqrt{x'(1-x')x(1-x)}(q'^{\mathrm{R}2}+q^{\mathrm{R}2})$
$--\,+-$	$+-\,++$
$--\,-+$	$-+\,++$
$--\,--$	$++\,++$

- The u-spinor mass is $\mathbf{m}$. The v-spinor mass is $\overline{\mathbf{m}}$.
- The q' and q are the transverse momenta to be integrated.