Scattering, Form Factors, and Exotics from Lattice QCD

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XSEDE
Extreme Science and Engineering Discovery Environment

Outline

- excited-states in lattice QCD
- finite-volume energies $\Rightarrow$ scattering phase shifts
- multi-hadron correlators and stochastic LapH method
- hadron resonance properties: masses, decay widths
- time-like pion form factor
- scalar glueball
- tetraquark operators and the $K_0^*(700), a_0(980)$
- string breaking
- current efforts: baryon-baryon scattering
- some recent lattice QCD highlights
- key points:
  - calculations at physical point now happening
  - can deal with disconnected diagrams

Excited states from correlation matrices

- energies from temporal correlations $C_{ij}(t) = \langle 0|\overline{O}_i(t) O_j(0)|0\rangle$
- in finite volume, energies are discrete (neglect wrap-around)

$$C_{ij}(t) = \sum_n Z_i^{(n)} Z_j^{(n)*}\, e^{-E_n t}, \qquad Z_j^{(n)} = \langle 0|\, O_j\, |n\rangle$$

- not practical to do fits using above form
- define new correlation matrix $\widetilde{C}(t)$ using a single rotation

$$\widetilde{C}(t) = U^\dagger\, C(\tau_0)^{-1/2}\, C(t)\, C(\tau_0)^{-1/2}\, U$$

- columns of U are eigenvectors of $C(\tau_0)^{-1/2}\, C(\tau_D)\, C(\tau_0)^{-1/2}$
- choose τ_0 and τ_D large enough so $\widetilde{C}(t)$ diagonal for $t > \tau_D$
- effective energies

$$\widetilde{m}_\alpha^{\rm eff}(t) = \frac{1}{\Delta t} \ln\left(\frac{\widetilde{C}_{\alpha\alpha}(t)}{\widetilde{C}_{\alpha\alpha}(t+\Delta t)}\right)$$

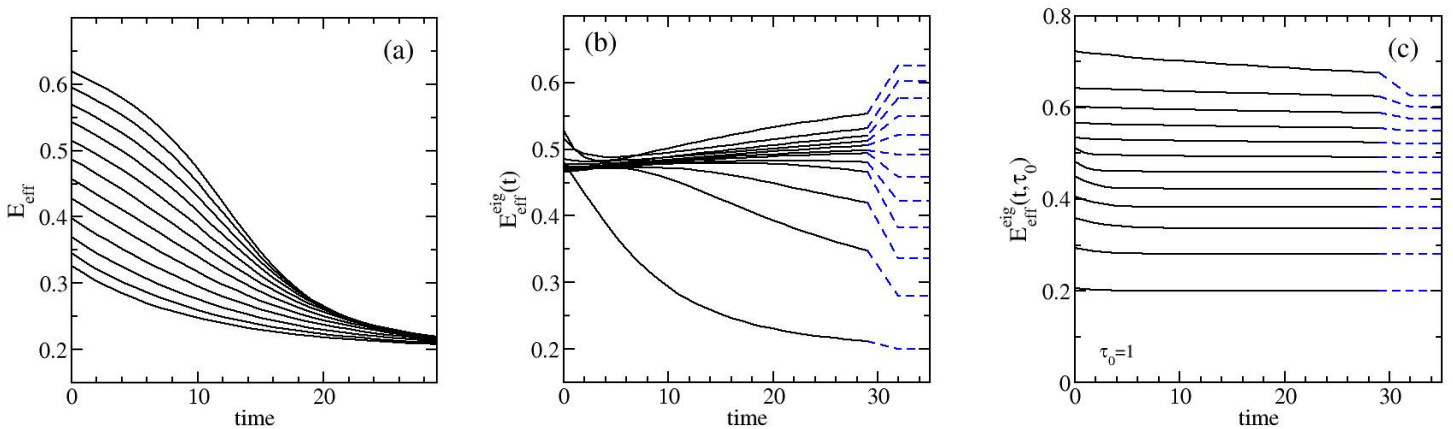
  tend to N lowest-lying stationary state energies in a channel

- 2-exponential fits to $\widetilde{C}_{\alpha\alpha}(t)$ yield energies E_α and overlaps $Z_j^{(n)}$

Correlator matrix toy model

- Example: 12×12 correlator matrix with $N_e = 200$ eigenstates

$$E_0 = 0.20, \qquad E_n = E_{n-1} + \frac{0.08}{\sqrt{n}}, \qquad Z_j^{(n)} = \frac{(-1)^{j+n}}{1 + 0.05(j-n)^2}.$$



- left: effective energies of diagonal elements of correlator matrix
- middle: effective energies of eigenvalues of $C(t)$
- right: effective energies of eigenvalues of $C(\tau_0)^{-1/2}\, C(t)\, C(\tau_0)^{-1/2}$ for $\tau_0 = 1$

Building blocks for single-hadron operators

- important to use good operators to see signal before noise growth
- building blocks: covariantly-displaced LapH-smeared quark fields
- stout links $\widetilde{U}_j(x)$
- Laplacian-Heaviside (LapH) smeared quark fields

$$\widetilde{\psi}_{a\alpha}(x) = \mathcal{S}_{ab}(x,y)\,\psi_{b\alpha}(y), \qquad \mathcal{S} = \Theta\left(\sigma_s^2 + \widetilde{\Delta}\right)$$

- 3d gauge-covariant Laplacian $\widetilde{\Delta}$ in terms of $\widetilde{U}$
- displaced quark fields:

$$q^A_{a\alpha j} = D^{(j)}\widetilde{\psi}^{(A)}_{a\alpha}, \qquad \overline{q}^A_{a\alpha j} = \overline{\widetilde{\psi}}^{(A)}_{a\alpha}\,\gamma_4\,D^{(j)\dagger}$$

- displacement $D^{(j)}$ is product of smeared links:

$$D^{(j)}(x,x') = \widetilde{U}_{j_1}(x)\,\widetilde{U}_{j_2}(x+d_2)\,\widetilde{U}_{j_3}(x+d_3)\ldots\widetilde{U}_{j_p}(x+d_p)\delta_{x',\,x+d_{p+1}}$$

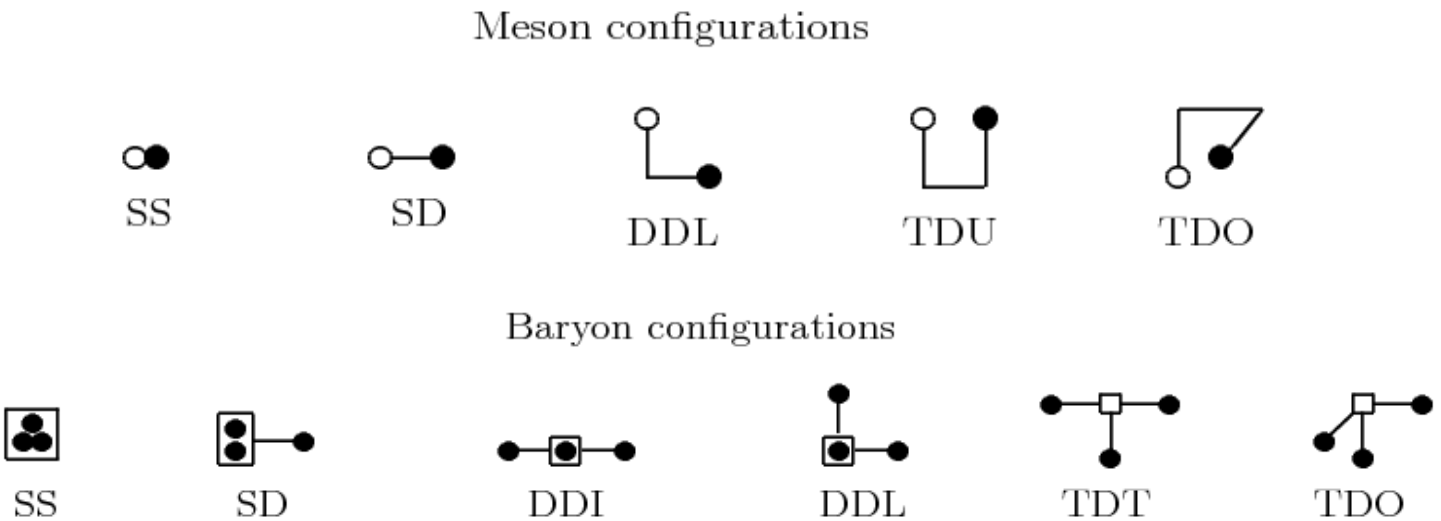
- to good approximation, LapH smearing operator is

$$\mathcal{S} = V_s V_s^\dagger$$

  - columns of matrix V_s are eigenvectors of $\widetilde{\Delta}$

Extended operators for single hadrons

- quark displacements build up orbital, radial structure



$$\overline{\Phi}^{AB}_{\alpha\beta}(\boldsymbol{p},t) = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot(\mathbf{x}+\frac{1}{2}(\boldsymbol{d}_\alpha+\boldsymbol{d}_\beta))}\delta_{ab}\,\overline{q}^B_{b\beta}(\boldsymbol{x},t)\,q^A_{a\alpha}(\boldsymbol{x},t)$$

$$\overline{\Phi}^{ABC}_{\alpha\beta\gamma}(\boldsymbol{p},t) = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\mathbf{x}}\varepsilon_{abc}\,\overline{q}^C_{c\gamma}(\boldsymbol{x},t)\,\overline{q}^B_{b\beta}(\boldsymbol{x},t)\,\overline{q}^A_{a\alpha}(\boldsymbol{x},t)$$

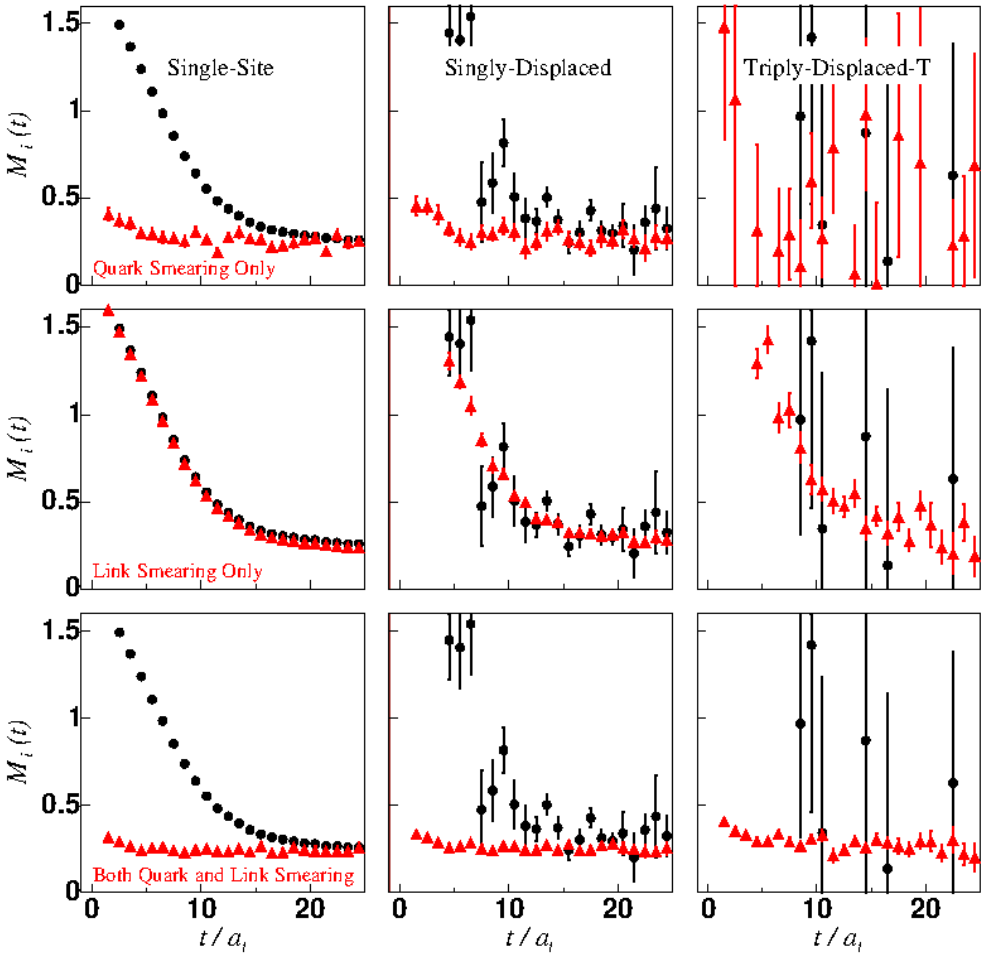
- group-theory projections onto irreps of lattice symmetry group

$$\overline{M}_l(t) = c^{(l)*}_{\alpha\beta}\,\overline{\Phi}^{AB}_{\alpha\beta}(t) \qquad \overline{B}_l(t) = c^{(l)*}_{\alpha\beta\gamma}\,\overline{\Phi}^{ABC}_{\alpha\beta\gamma}(t)$$

- definite momentum $\boldsymbol{p}$, irreps of little group of $\boldsymbol{p}$

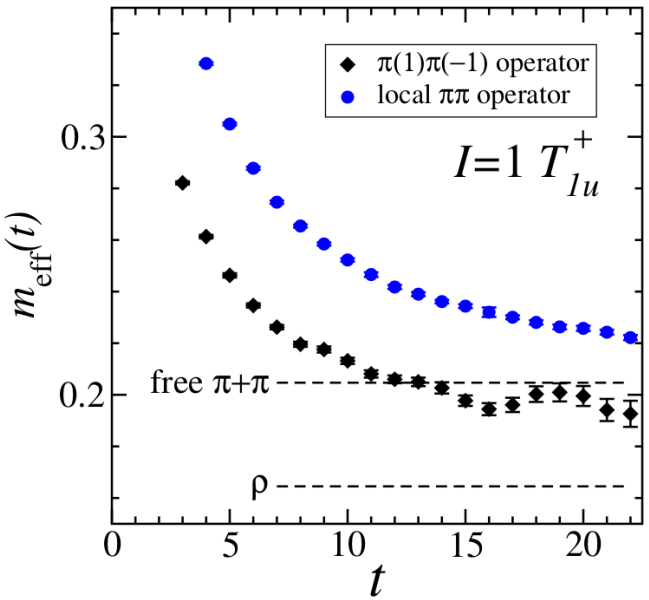
Importance of smeared fields

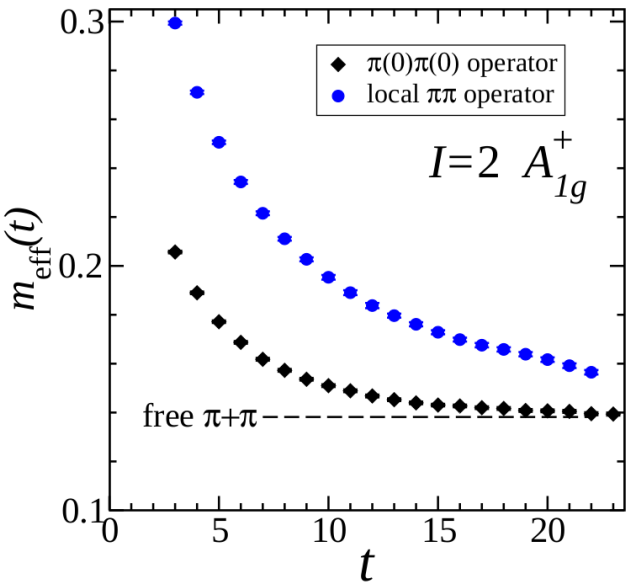
- effective masses of 3 selected nucleon operators shown
- noise reduction of displaced-operators from link smearing $n_\rho \rho = 2.5, n_\rho = 16$
- quark-field smearing $\sigma_s = 4.0, n_\sigma = 32$ reduces excited-state contamination



Two-hadron operators

- comparison of $\pi(\boldsymbol{k})\pi(-\boldsymbol{k})$ and localized $\sum_{\boldsymbol{x}} \pi(\boldsymbol{x})\pi(\boldsymbol{x})$ operators





- important to use superposition of products of single-hadron operators of definite momenta
- efficient construction, generalizes to three or more hadrons

Quark propagation

- quark propagator Q is inverse D^{-1} of Dirac matrix
  - rows/columns involve lattice site, spin, color
  - very large $N_{\rm tot} \times N_{\rm tot}$ matrix for each flavor

$$N_{\rm tot} = N_{\rm site} N_{\rm spin} N_{\rm color}$$

  - for $64^3 \times 128$ lattice, $N_{\rm tot} \sim 400$ million
- not feasible to compute (or store) all elements of D^{-1}
- point-to-all trick for local operators: use translation invariance

$$\sum_{\mathbf{y}} \sum_{\mathbf{x}} Q^{(a)}(\mathbf{y}, t_f | \mathbf{x}, t_i)\, Q^{(b)}(\mathbf{y}, t_f | \mathbf{x}, t_i) \cdots \longrightarrow \sum_{\mathbf{y}} Q^{(a)}(\mathbf{y}, t_f | \mathbf{x}_0, t_i)\, Q^{(b)}(\mathbf{y}, t_f | \mathbf{x}_0, t_i) \ldots$$

- cannot use this trick for good multi-hadron operators

$$\sum_{\mathbf{y}_1, \mathbf{y}_2, \ldots} \sum_{\mathbf{x}_1, \mathbf{x}_2, \ldots} Q^{(a)}(\mathbf{y}_1, t_f | \mathbf{x}_1, t_i)\, Q^{(b)}(\mathbf{y}_2, t_f | \mathbf{x}_2, t_i) \ldots$$

- our solution: the stochastic LapH method!
- other methods

Stochastic estimation of quark propagators

- do not need exact inverse of Dirac matrix $D[U]$
- introduce Z_4 noise vectors η in the LapH subspace

$$\eta_{\alpha k}(t), \qquad t = \text{time},\ \alpha = \text{spin},\ k = \text{eigenvector number}$$

- solve $D[U]X^{(r)} = \eta^{(r)}$ for each of N_R noise vectors $\eta^{(r)}$, then obtain a Monte Carlo estimate of all elements of D^{-1}

$$D^{-1}_{ij} \approx \frac{1}{N_R}\sum_{r=1}^{N_R} X^{(r)}_i \eta^{(r)*}_j$$

- variance reduction using noise dilution
- dilution introduces projectors $P^{(a)}$, then define

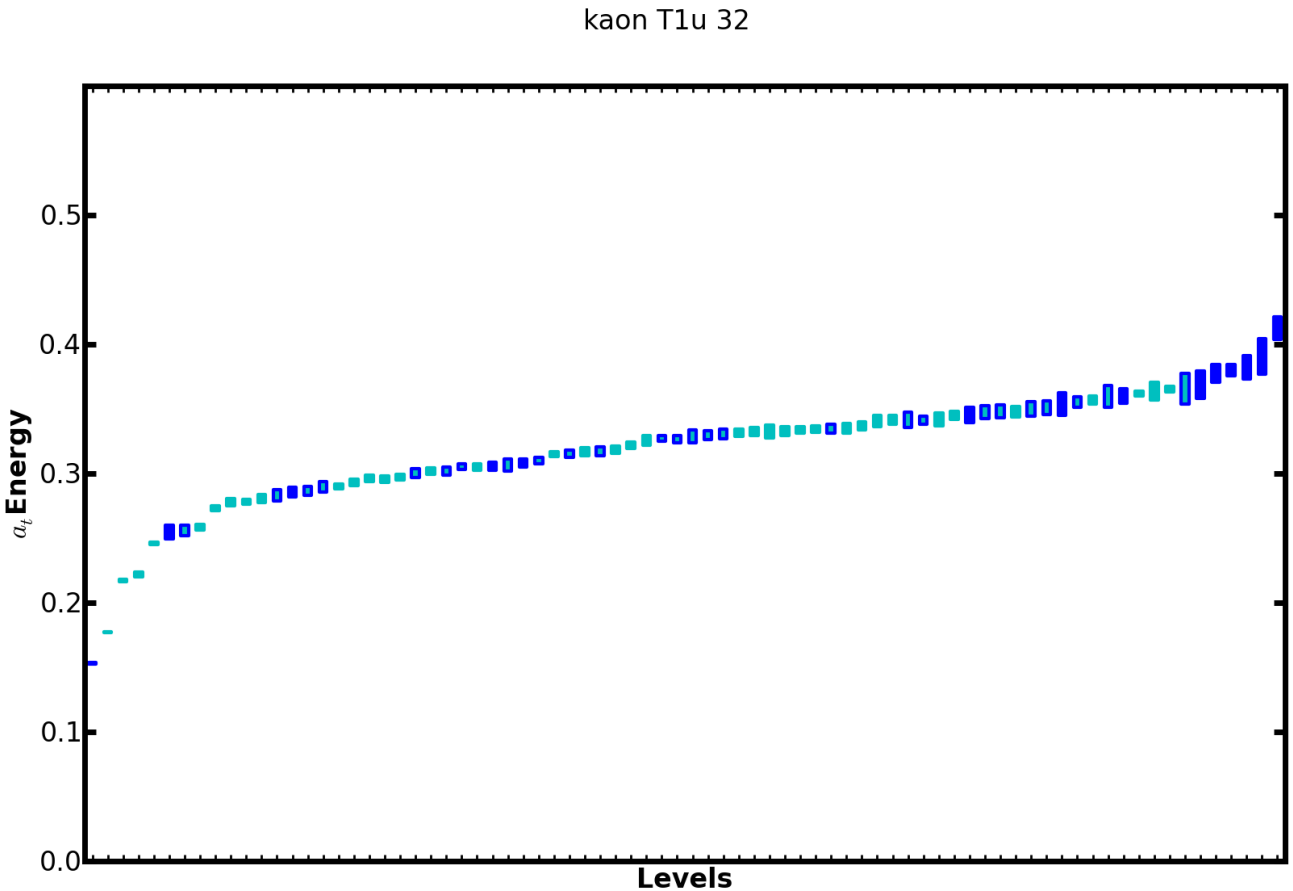
$$\eta^{[a]} = P^{(a)}\eta, \qquad X^{[a]} = D^{-1}\eta^{[a]}$$

to obtain Monte Carlo estimate with drastically reduced variance

$$D^{-1}_{ij} \approx \frac{1}{N_R}\sum_{r=1}^{N_R}\sum_{a} X^{(r)[a]}_i \eta^{(r)[a]*}_j$$

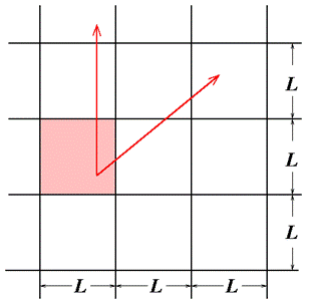
Bosonic $I = \frac{1}{2},\ S = 1,\ T_{1u}$ channel

- finite-volume stationary-state energies: “staircase” plot
- $32^3 \times 256$ lattice for $m_\pi \sim 240$ MeV
- use of single- and two-meson operators only
- blue: levels of max ovelaps with SH optimized operators



Quantum numbers in toroidal box

- periodic boundary conditions in cubic box
  - not all directions equivalent $\Rightarrow$ using J^{PC} is wrong!!



- label stationary states of QCD in a periodic box using irreps of cubic space group even in continuum limit
  - zero momentum states: little group O_h

$$A_{1a}, A_{2ga}, E_a, T_{1a}, T_{2a}, \quad G_{1a}, G_{2a}, H_a, \qquad a = g, u$$

  - on-axis momenta: little group C_{4v}

$$A_1, A_2, B_1, B_2, E, \quad G_1, G_2$$

  - planar-diagonal momenta: little group C_{2v}

$$A_1, A_2, B_1, B_2, \quad G_1, G_2$$

  - cubic-diagonal momenta: little group C_{3v}

$$A_1, A_2, E, \quad F_1, F_2, G$$

- include G parity in some meson sectors (superscript $+$ or $-$)

Spin content of cubic box irreps

- numbers of occurrences of Λ irreps in J subduced

J	A_1	A_2	E	T_1	T_2
0	1	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	1
3	0	1	0	1	1
4	1	0	1	1	1
5	0	0	1	2	1
6	1	1	1	1	2
7	0	1	1	2	2

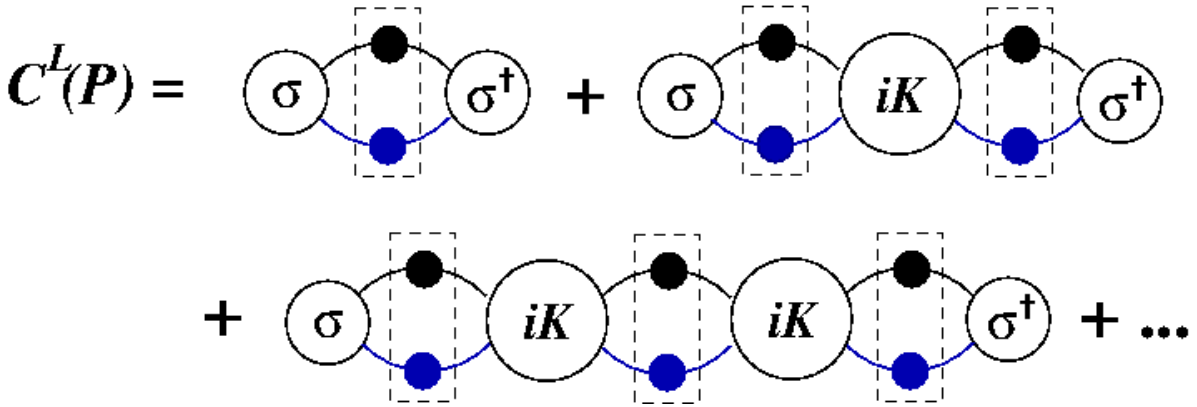
J	G_1	G_2	H	J	G_1	G_2	H
$\frac{1}{2}$	1	0	0	$\frac{9}{2}$	1	0	2
$\frac{3}{2}$	0	0	1	$\frac{11}{2}$	1	1	2
$\frac{5}{2}$	0	1	1	$\frac{13}{2}$	1	2	2
$\frac{7}{2}$	1	1	1	$\frac{15}{2}$	1	1	3

Scattering phase shifts in lattice QCD timeline

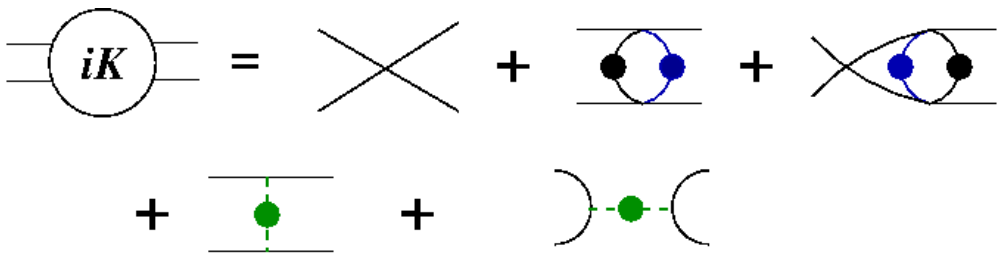
- Lüscher 1984: energies of multi-hadron stationary states in finite volume can yield scattering phase shifts
- Rummukainen and Gottlieb 1995: nonzero total momenta
- Kim, Sachrajda, and Sharpe 2005: revisited
- explosion of papers since then
- Briceno 2014: generalized to arbitrary spin, multiple channels

Two-particle correlator in finite-volume

- correlator of two-particle operator σ in finite volume



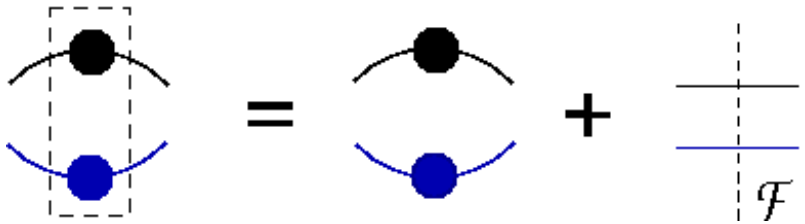
- Bethe-Salpeter kernel



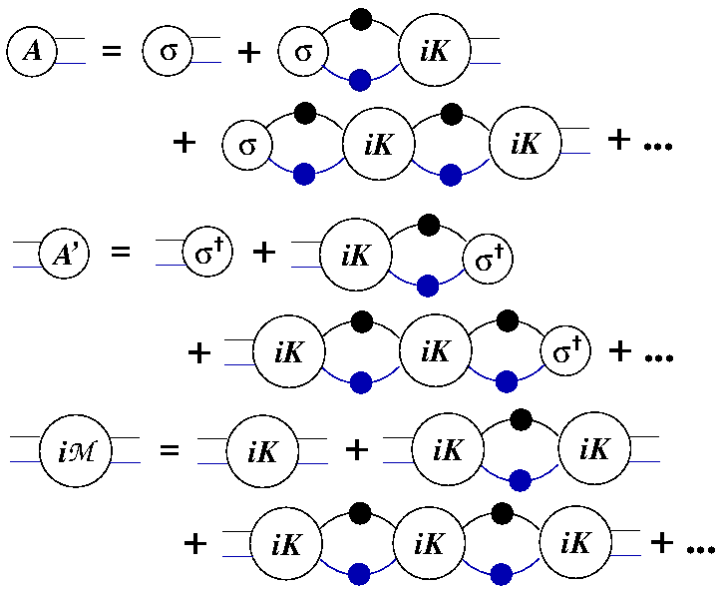
- $C^\infty(P)$ has branch cuts where two-particle thresholds begin
- momentum quantization in finite volume: cuts $\rightarrow$ series of poles
- C^L poles: two-particle energy spectrum of finite volume theory

Corrections from finite momentum sums

- finite-volume momentum sum is infinite-volume integral plus correction $\mathcal{F}$

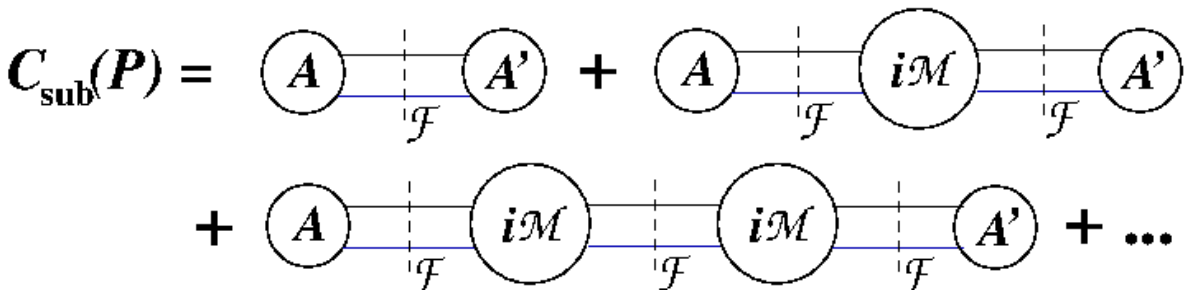


- define the following quantities: A, A', invariant scattering amplitude $i\mathcal{M}$



Quantization condition

- subtracted correlator $C_{\rm sub}(P) = C^L(P) - C^\infty(P)$ given by



- sum geometric series

$$C_{\rm sub}(P) = A\,\mathcal{F}(1 - i\mathcal{M}\mathcal{F})^{-1}\,A'.$$

- poles of $C_{\rm sub}(P)$ are poles of $C^L(P)$ from $\det(1 - i\mathcal{M}\mathcal{F}) = 0$
- key tool: for $g_c(\boldsymbol{p})$ spatially contained and regular

$$\frac{1}{L^3}\sum_{\boldsymbol{p}} g_c(\boldsymbol{p}) = \int \frac{d^3k}{(2\pi)^3}\, g_c(\mathbf{k}) + O(e^{-mL})$$

$$\frac{1}{L^3}\sum_{\boldsymbol{p}} \frac{g_c(\boldsymbol{p}^2)}{(\boldsymbol{p}^2 - a^2)} = \frac{1}{L^3}\sum_{\boldsymbol{p}} \frac{g_c(a^2)}{(\boldsymbol{p}^2 - a^2)} + \int \frac{d^3k}{(2\pi)^3} \frac{g_c(\boldsymbol{p}^2) - g(a^2)}{(\boldsymbol{p}^2 - a^2)} + O(e^{-mL})$$

Kinematics

- work in spatial L^3 volume with periodic b.c.
- total momentum $\boldsymbol{P} = (2\pi/L)\boldsymbol{d}$, where $\boldsymbol{d}$ vector of integers
- calculate lab-frame energy E of two-particle interacting state in lattice QCD
- boost to center-of-mass frame by defining:

$$E_{\rm cm} = \sqrt{E^2 - \boldsymbol{P}^2}, \qquad \gamma = \frac{E}{E_{\rm cm}},$$

- assume N_d channels
- particle masses m_{1a}, m_{2a} and spins s_{1a}, s_{2a} of particle 1 and 2
- for each channel, can calculate

$$\begin{aligned} \boldsymbol{q}^2_{{\rm cm},a} &= \frac{1}{4}E^2_{\rm cm} - \frac{1}{2}(m^2_{1a} + m^2_{2a}) + \frac{(m^2_{1a} - m^2_{2a})^2}{4E^2_{\rm cm}}, \\ u^2_a &= \frac{L^2\boldsymbol{q}^2_{{\rm cm},a}}{(2\pi)^2}, \qquad \boldsymbol{s}_a = \left(1 + \frac{(m^2_{1a} - m^2_{2a})}{E^2_{\rm cm}}\right)\boldsymbol{d} \end{aligned}$$

Quantization condition re-expressed

- E related to S matrix (and phase shifts) by

$$\det[1 + F^{(\boldsymbol{P})}(S-1)] = 0$$

- F matrix in $JLSa$ basis states given by

$$\langle J' m_{J'} L' S' a' | F^{(\boldsymbol{P})} | J m_J L S a \rangle = \delta_{a'a}\delta_{S'S} \frac{1}{2}\Big\{ \delta_{J'J}\delta_{m_{J'}m_J}\delta_{L'L} + \langle J' m_{J'} | L' m_{L'} S m_S \rangle \langle L m_L S m_S | J m_J \rangle W^{(\boldsymbol{P}a)}_{L' m_{L'};\, L m_L} \Big\}$$

- total ang mom J, J', orbital L, L', spin S, S', channels a, a'
- W given by

$$-iW^{(\boldsymbol{P}a)}_{L' m_{L'};\, L m_L} = \sum_{l=|L'-L|}^{L'+L} \sum_{m=-l}^{l} \frac{\mathcal{Z}_{lm}(\boldsymbol{s}_a, \gamma, u_a^2)}{\pi^{3/2}\gamma u_a^{l+1}} \sqrt{\frac{(2L'+1)(2l+1)}{(2L+1)}} \times \langle L'0, l0 | L0 \rangle \langle L' m_{L'}, lm | L m_L \rangle .$$

- above expressions apply for both distinguishable and indistinguishable particles

RGL shifted zeta functions

- compute Rummukainen-Gottlieb-Lüscher (RGL) shifted zeta functions $\mathcal{Z}_{lm}$ using

$$\mathcal{Z}_{lm}(\boldsymbol{s},\gamma,u^2) = \sum_{\boldsymbol{n}\in\mathbb{Z}^3} \frac{\mathcal{Y}_{lm}(\boldsymbol{z})}{(\boldsymbol{z}^2-u^2)} e^{-\Lambda(\boldsymbol{z}^2-u^2)} + \delta_{l0}\frac{\gamma\pi}{\sqrt{\Lambda}} F_0(\Lambda u^2)$$

$$+\frac{i^l\gamma}{\Lambda^{l+1/2}} \int_0^1 dt \left(\frac{\pi}{t}\right)^{l+3/2} e^{\Lambda t u^2} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^3 \\ \boldsymbol{n}\neq 0}} e^{\pi i \boldsymbol{n}\cdot\boldsymbol{s}} \mathcal{Y}_{lm}(\mathbf{w})\, e^{-\pi^2\mathbf{w}^2/(t\Lambda)}$$

- where

$$\boldsymbol{z} = \boldsymbol{n} - \gamma^{-1}\left[\tfrac{1}{2} + (\gamma-1)s^{-2}\boldsymbol{n}\cdot\boldsymbol{s}\right]\boldsymbol{s},$$

$$\mathbf{w} = \boldsymbol{n} - (1-\gamma)s^{-2}\boldsymbol{s}\cdot\boldsymbol{n}\boldsymbol{s}, \qquad \mathcal{Y}_{lm}(\mathbf{x}) = |\mathbf{x}|^l\, Y_{lm}(\widehat{\mathbf{x}})$$

$$F_0(x) = -1 + \frac{1}{2}\int_0^1 dt\, \frac{e^{tx}-1}{t^{3/2}}$$

- choose $\Lambda \approx 1$ for convergence of the summation
- integral done Gauss-Legendre quadrature
- $F_0(x)$ given in terms of Dawson or erf function

K matrix

- quantization condition relates single energy E to entire S-matrix
- cannot solve for S-matrix (except single channel, single wave)
- approximate S-matrix with functions depending on handful of fit parameters
- obtain estimates of fit parameters using many energies
- easier to parametrize Hermitian matrix than unitary matrix
- introduce K-matrix (Wigner 1946)

$$S = (1 + iK)(1 - iK)^{-1} = (1 - iK)^{-1}(1 + iK)$$

- Hermiticity of K-matrix ensures unitarity of S-matrix
- with time reversal invariance, K-matrix must be real and symmetric

K matrix

- multichannel effective range expansion (Ross 1961)

$$K^{-1}_{L'S'a';\,LSa}(E) = q_{a'}^{-L'-\frac{1}{2}}\ \widetilde{K}^{-1}_{L'S'a';\,LSa}(E_{\rm cm})\ q_a^{-L-\frac{1}{2}},$$

- quantization condition can be written

$$\det(1 - B^{(\boldsymbol{P})}\widetilde{K}) = \det(1 - \widetilde{K}B^{(\boldsymbol{P})}) = 0$$

- we define the box matrix by

$$\langle J'm_{J'}L'S'a'|\ B^{(\boldsymbol{P})}\ |Jm_JLSa\rangle = -i\delta_{a'a}\delta_{S'S}\ u_a^{L'+L+1}\ W^{(\boldsymbol{P}a)}_{L'm_{L'};\,Lm_L}$$
$$\times\langle J'm_{J'}|L'm_{L'},Sm_S\rangle\langle Lm_L,Sm_S|Jm_J\rangle$$

- box matrix is Hermitian for u_a^2 real
- quantization condition can also be expressed as

$$\det(\widetilde{K}^{-1} - B^{(\boldsymbol{P})}) = 0$$

- these determinants are real

Block diagonalization

- quantization condition involves determinant of infinite matrix
- make practical by (a) transforming to a block-diagonal basis and (b) truncating in orbital angular momentum
- block-diagonal basis

$$|\Lambda\lambda n J L S a\rangle = \sum_{m_J} c_{m_J}^{J(-1)^L;\,\Lambda\lambda n} |J m_J L S a\rangle$$

- little group irrep Λ, irrep row λ, occurrence index n
- transformation coefficients depend on J and $(-1)^L$, not on S, a
- replaces m_J by (Λ, λ, n)
- group theoretical projections with Gram-Schmidt used to obtain coefficients
- use notation and irrep matrices from PRD 88, 014511 (2013)

Box and $\widetilde{K}$ matrices in block diagonal basis

- in block-diagonal basis, box matrix has form

$$\langle \Lambda' \lambda' n' J' L' S' a' |\, B^{(\boldsymbol{P})} \,| \Lambda \lambda n J L S a \rangle = \delta_{\Lambda' \Lambda} \delta_{\lambda' \lambda} \delta_{S' S} \delta_{a' a}\, B^{(\boldsymbol{P} \Lambda_B S a)}_{J' L' n';\; J L n}(E)$$

- $\widetilde{K}$-matrix for $(-1)^{L+L'} = 1$ has form

$$\langle \Lambda' \lambda' n' J' L' S' a' |\, \widetilde{K} \,| \Lambda \lambda n J L S a \rangle = \delta_{\Lambda' \Lambda} \delta_{\lambda' \lambda} \delta_{n' n} \delta_{J' J}\, \mathcal{K}^{(J)}_{L' S' a';\; L S a}(E_{\rm cm})$$

- $(-1)^{L+L'} = 1 \Rightarrow \eta^{P\prime}_{1a'} \eta^{P\prime}_{2a'} = \eta^{P}_{1a} \eta^{P}_{2a}$, always applies in QCD
- Λ is irrep for K-matrix, need Λ_B for box matrix
- when $\eta^{P}_{1a} \eta^{P}_{2a} = 1$, then $\Lambda_B = \Lambda$

$\boldsymbol{d}$	LG	Λ_B relationship to Λ when $\eta^{P}_{1a} \eta^{P}_{2a} = -1$
$(0,0,0)$	O_h	Subscript $g \leftrightarrow u$
$(0,0,n)$	C_{4v}	$A_1 \leftrightarrow A_2;\ B_1 \leftrightarrow B_2$; E, G_1, G_2 stay same
$(0,n,n)$	C_{2v}	$A_1 \leftrightarrow A_2;\ B_1 \leftrightarrow B_2$; G stays same
(n,n,n)	C_{3v}	$A_1 \leftrightarrow A_2;\ F_1 \leftrightarrow F_2$; E, G stay same

- see PRD 88, 014511 (2013) for notation

K matrix parametrizations

- $\widetilde{K}$ matrix in block diagonal basis

$$\langle \Lambda' \lambda' n' J' L' S' a' |\, \widetilde{K} \,| \Lambda \lambda n J L S a \rangle = \delta_{\Lambda' \Lambda} \delta_{\lambda' \lambda} \delta_{n' n} \delta_{J' J}\, \mathcal{K}^{(J)}_{L' S' a';\, L S a}(E_{\rm cm})$$

$$\langle \Lambda' \lambda' n' J' L' S' a' |\, \widetilde{K}^{-1} \,| \Lambda \lambda n J L S a \rangle = \delta_{\Lambda' \Lambda} \delta_{\lambda' \lambda} \delta_{n' n} \delta_{J' J}\, \mathcal{K}^{(J)-1}_{L' S' a';\, L S a}(E_{\rm cm})$$

- common parametrization

$$\mathcal{K}^{(J)-1}_{\alpha\beta}(E_{\rm cm}) = \sum_{k=0}^{N_{\alpha\beta}} c^{(Jk)}_{\alpha\beta} E^k_{\rm cm}$$

- α, β compound indices for (L, S, a)
- another common parametrization

$$\mathcal{K}^{(J)}_{\alpha\beta}(E_{\rm cm}) = \sum_p \frac{g^{(Jp)}_\alpha g^{(Jp)}_\beta}{E^2_{\rm cm} - m^2_{Jp}} + \sum_k d^{(Jk)}_{\alpha\beta} E^k_{\rm cm},$$

- Lorentz invariant form using $E_{\rm cm} = \sqrt{s}$
- Mandelstam variable $s = (p_1 + p_2)^2$, with p_j four-momentum of particle j

Box matrix elements

- have obtained expressions for $B^{(\boldsymbol{P}\Lambda_B Sa)}_{J'L'n';\ JLn}(E)$ for
- $L \leq 6$, $S \leq 2$ with $\boldsymbol{P} = (0,0,0), (0,0,p)$, $p > 0$
- $L \leq 6$, $S \leq \frac{3}{2}$ with $\boldsymbol{P} = (0,p,p), (p,p,p)$, $p > 0$
- in tables that follow, we define

  R_{lm} is short hand for $(\gamma\pi^{3/2}u_a^{l+1})^{-1}\mathrm{Re}\ \mathcal{Z}_{lm}(\boldsymbol{s}_a, \gamma, u_a^2)$

  I_{lm} is short hand for $(\gamma\pi^{3/2}u_a^{l+1})^{-1}\mathrm{Im}\ \mathcal{Z}_{lm}(\boldsymbol{s}_a, \gamma, u_a^2)$
- C++ software available on github: TwoHadronsInBox

Box matrix elements $P = 0,\ S = 0$

J'	L'	n'	J	L	n	$u_a^{-(L'+L+1)}\ B$
						$\Lambda_B = A_{1g}$
0	0	1	0	0	1	R_{00}
0	0	1	4	4	1	$\frac{2\sqrt{21}}{7} R_{40}$
0	0	1	6	6	1	$-2\sqrt{2} R_{60}$
4	4	1	4	4	1	$R_{00} + \frac{108}{143} R_{40} + \frac{80\sqrt{13}}{143} R_{60} + \frac{560\sqrt{17}}{2431} R_{80}$
4	4	1	6	6	1	$-\frac{40\sqrt{546}}{1001} R_{40} + \frac{42\sqrt{42}}{187} R_{60} - \frac{224\sqrt{9282}}{46189} R_{80} - \frac{1008\sqrt{26}}{4199} R_{10,0}$
6	6	1	6	6	1	$R_{00} - \frac{126}{187} R_{40} - \frac{160\sqrt{13}}{3553} R_{60} + \frac{840\sqrt{17}}{3553} R_{80} - \frac{2016\sqrt{21}}{7429} R_{10,0}$ $+ \frac{30492}{37145} R_{12,0} - \frac{1848\sqrt{1001}}{37145} R_{12,4}$
						$\Lambda_B = A_{2g}$
6	6	1	6	6	1	$R_{00} + \frac{6}{17} R_{40} - \frac{160\sqrt{13}}{323} R_{60} - \frac{40\sqrt{17}}{323} R_{80} - \frac{2592\sqrt{21}}{7429} R_{10,0}$ $+ \frac{1980}{7429} R_{12,0} + \frac{264\sqrt{1001}}{7429} R_{12,4}$
						$\Lambda_B = A_{2u}$
3	3	1	3	3	1	$R_{00} - \frac{12}{11} R_{40} + \frac{80\sqrt{13}}{143} R_{60}$

Box matrix elements $P = 0,\ S = 0$

J'	L'	n'	J	L	n	$u_a^{-(L'+L+1)}\ B$
						$\Lambda_B = E_g$
2	2	1	2	2	1	$R_{00} + \frac{6}{7} R_{40}$
2	2	1	4	4	1	$-\frac{40\sqrt{3}}{77} R_{40} - \frac{30\sqrt{39}}{143} R_{60}$
2	2	1	6	6	1	$\frac{30\sqrt{910}}{1001} R_{40} + \frac{4\sqrt{70}}{55} R_{60} + \frac{8\sqrt{15470}}{1105} R_{80}$
4	4	1	4	4	1	$R_{00} + \frac{108}{1001} R_{40} - \frac{64\sqrt{13}}{143} R_{60} + \frac{392\sqrt{17}}{2431} R_{80}$
4	4	1	6	6	1	$-\frac{8\sqrt{2730}}{1001} R_{40} - \frac{18\sqrt{210}}{187} R_{60} - \frac{128\sqrt{46410}}{46189} R_{80}$ $-\frac{1512\sqrt{130}}{20995} R_{10,0}$
6	6	1	6	6	1	$R_{00} + \frac{114}{187} R_{40} + \frac{480\sqrt{13}}{3553} R_{60} + \frac{280\sqrt{17}}{3553} R_{80} + \frac{1152\sqrt{21}}{7429} R_{10,0}$ $+\frac{30492}{37145} R_{12,0} + \frac{264\sqrt{1001}}{37145} R_{12,4}$
						$\Lambda_B = E_u$
5	5	1	5	5	1	$R_{00} - \frac{6}{13} R_{40} + \frac{32\sqrt{13}}{221} R_{60} - \frac{672\sqrt{17}}{4199} R_{80} + \frac{1152\sqrt{21}}{4199} R_{10,0}$
						$\Lambda_B = T_{1g}$
4	4	1	4	4	1	$R_{00} + \frac{54}{143} R_{40} - \frac{4\sqrt{13}}{143} R_{60} - \frac{448\sqrt{17}}{2431} R_{80}$
4	4	1	6	6	1	$-\frac{12\sqrt{65}}{143} R_{40} + \frac{42\sqrt{5}}{187} R_{60} + \frac{112\sqrt{1105}}{46189} R_{80} + \frac{576\sqrt{1365}}{20995} R_{10,0}$
6	6	1	6	6	1	$R_{00} - \frac{96}{187} R_{40} - \frac{80\sqrt{13}}{3553} R_{60} + \frac{120\sqrt{17}}{3553} R_{80} + \frac{624\sqrt{21}}{7429} R_{10,0}$ $-\frac{26136}{37145} R_{12,0} + \frac{1584\sqrt{1001}}{37145} R_{12,4}$

Box matrix elements $P = (2\pi/L)(0, n, n),\ S = \frac{1}{2}$

J'	L'	n'	J	L	n	$u_a^{-(L'+L+1)}\ B$
						$\Lambda_B = G$ (partial)
$\frac{5}{2}$	2	2	$\frac{9}{2}$	5	4	$-\frac{3\sqrt{105}}{308} iR_{30} - \frac{13\sqrt{14}}{924} iR_{32} - \frac{7\sqrt{165}}{286} iR_{50} + \frac{95\sqrt{154}}{3003} iR_{52} - \frac{25\sqrt{462}}{2002} iR_{54} + \frac{915}{2288} iR_{70} + \frac{375\sqrt{21}}{16016} iR_{72} - \frac{675\sqrt{462}}{16016} iR_{74} + \frac{15\sqrt{3003}}{2288} iR_{76}$
$\frac{5}{2}$	2	2	$\frac{9}{2}$	5	5	$-\frac{23\sqrt{30}}{924} R_{30} - \frac{95}{462} R_{32} - \frac{2\sqrt{2310}}{3003} R_{50} + \frac{2\sqrt{11}}{429} R_{52} + \frac{16\sqrt{33}}{429} R_{54} + \frac{135\sqrt{14}}{2288} R_{70} + \frac{435\sqrt{6}}{2288} R_{72} + \frac{105\sqrt{33}}{1144} R_{74} + \frac{45\sqrt{858}}{2288} R_{76}$
$\frac{5}{2}$	2	2	$\frac{11}{2}$	5	1	$\frac{\sqrt{105}}{13} R_{54} - \frac{\sqrt{105}}{65} R_{74} - \frac{\sqrt{2730}}{455} R_{76}$
$\frac{5}{2}$	2	2	$\frac{11}{2}$	5	2	$-\frac{5\sqrt{35}}{77} R_{32} + \frac{10\sqrt{385}}{1001} R_{52} - \frac{\sqrt{1155}}{1001} R_{54} - \frac{5\sqrt{210}}{2002} R_{72} + \frac{2\sqrt{1155}}{715} R_{74} + \frac{3\sqrt{30030}}{1430} R_{76}$
$\frac{5}{2}$	2	2	$\frac{11}{2}$	5	3	$-\frac{5\sqrt{70}}{231} R_{30} + \frac{10\sqrt{21}}{231} R_{32} + \frac{10\sqrt{110}}{429} R_{50} + \frac{2\sqrt{231}}{273} R_{52} - \frac{\sqrt{77}}{13} R_{54} - \frac{5\sqrt{6}}{143} R_{70} + \frac{27\sqrt{14}}{1001} R_{72} - \frac{3\sqrt{77}}{143} R_{74}$
$\frac{5}{2}$	2	2	$\frac{11}{2}$	5	4	$\frac{5\sqrt{7}}{11} R_{32} + \frac{8\sqrt{77}}{143} R_{52} - \frac{9\sqrt{231}}{1001} R_{54} - \frac{17\sqrt{42}}{286} R_{72} - \frac{6\sqrt{231}}{1001} R_{74} - \frac{5\sqrt{6006}}{2002} R_{76}$
$\frac{5}{2}$	2	2	$\frac{11}{2}$	5	5	$\frac{5\sqrt{35}}{33} R_{30} + \frac{5\sqrt{42}}{231} R_{32} - \frac{7\sqrt{55}}{429} R_{50} - \frac{\sqrt{462}}{3003} R_{52} + \frac{10\sqrt{154}}{1001} R_{54} - \frac{42\sqrt{3}}{143} R_{70} - \frac{6\sqrt{7}}{1001} R_{72} - \frac{15\sqrt{154}}{1001} R_{74}$
$\frac{5}{2}$	2	2	$\frac{11}{2}$	5	6	$\frac{50}{231} iR_{30} + \frac{5\sqrt{30}}{77} iR_{32} + \frac{5\sqrt{77}}{429} iR_{50} - \frac{3\sqrt{330}}{143} iR_{52} + \frac{4\sqrt{105}}{715} iR_{70} - \frac{192\sqrt{5}}{715} iR_{72}$

Fitting: spectrum method

- adjust K-matrix parameters to reproduce energies from lattice QCD
- choose $E_{\mathrm{cm},k}$ as observables
- model predictions by solving quantization for κ_j parameters
- problems:
  - root finding difficult, many computations of RGL zeta functions
  - ambiguity mapping model energies to observed energies
  - model predictions depend on observables $m_{1a},\ m_{2a},\ L,\ \xi$ so must recompute covariance during minimization
- "Lagrange multiplier" trick removes obs. dependence in model
  - include $m_{1a},\ m_{2a},\ L,\ \xi$ as both observables and model parameters
- observations

  Observations R_i: $\{E_{\mathrm{cm},k}^{(\mathrm{obs})},\ m_j^{(\mathrm{obs})},\ L^{(\mathrm{obs})},\ \xi^{(\mathrm{obs})}\},$
- model parameters

  Model fit parameters α_k: $\{\ \kappa_i,\ m_j^{(\mathrm{model})},\ L^{(\mathrm{model})},\ \xi^{(\mathrm{model})}\ \},$

Fitting: spectrum method (con't)

- residuals

$$r_k = \begin{cases} E^{(\mathrm{obs})}_{\mathrm{cm},k} - E^{(\mathrm{model})}_{\mathrm{cm},k}, & (k = 1, \ldots, N_E), \\ m^{(\mathrm{obs})}_{k'} - m^{(\mathrm{model})}_{k'}, & (k = k' + N_E,\ k' = 1, \ldots, N_p), \\ L^{(\mathrm{obs})} - L^{(\mathrm{model})}, & (k = N_E + N_p + 1), \\ \xi^{(\mathrm{obs})} - \xi^{(\mathrm{model})}, & (k = N_E + N_p + 2). \end{cases}$$

- compute $E^{(\mathrm{model})}_{\mathrm{cm},k}$ using only model parameters
- emphasize $E^{(\mathrm{model})}_{\mathrm{cm},k}$ very difficult to compute

Fitting: determinant residual method

- introduce quantization determinant as residual
- better to use function of matrix A with real parameter μ:

$$\Omega(\mu, A) \equiv \frac{\det(A)}{\det[(\mu^2 + AA^\dagger)^{1/2}]}$$

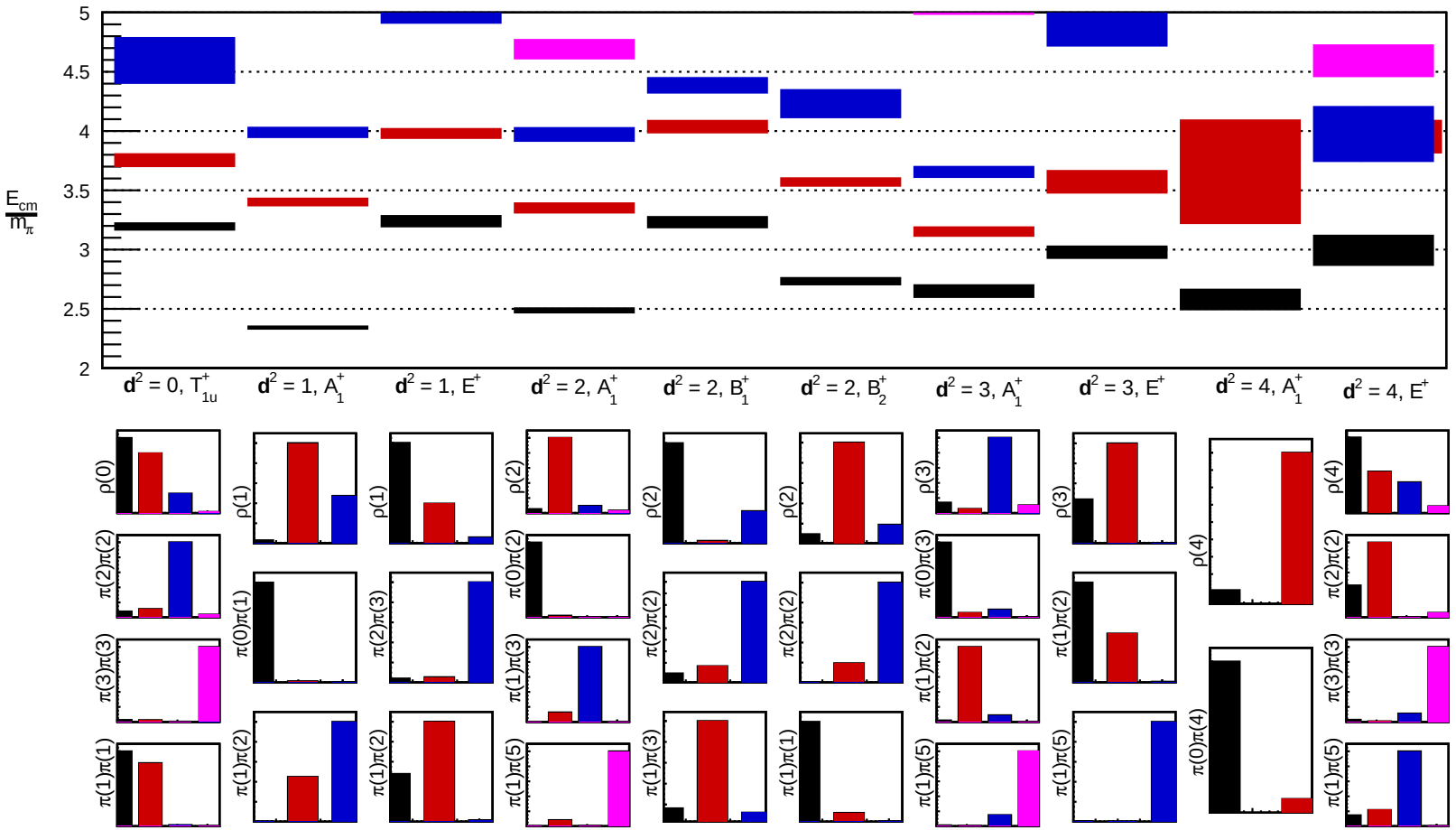
- evaluated using only eigenvalues; role of μ to suppress contributions from irrelevant higher-lying eigenvalues
- residuals

$$r_k = \Omega\Big(\mu, 1 - B^{(\boldsymbol{P})}(E^{(\mathrm{obs})}_{\mathrm{cm},k})\ \widetilde{K}(E^{(\mathrm{obs})}_{\mathrm{cm},k})\Big), \qquad (k = 1, \ldots, N_E),$$

- use only observed energies, particle masses, lattice size, anisotropy
- advantage: model predictions do not need root finding or RGL zeta computations
- model depends on observables, so covariance must be recomputed as κ_j parameters adjusted during minimization
- covariance recomputation still much simpler than root finding required in spectrum method

$\pi\pi$ $I = 1$ energies in finite volume

- finite volume energies $32^3 \times 256$ lattice, $m_\pi \approx 240$ MeV



Decay width of ρ

- applied to $I = 1$ $\rho \to \pi\pi$ system NPB 910, 842 (2016)
- included $L = 1, 3, 5$ partial waves in NPB 924, 477 (2017)
- large $32^3 \times 256$ anisotropic lattice, $m_\pi \approx 240$ MeV
- fit forms (first ever inclusion of $L = 5$ in lattice QCD):

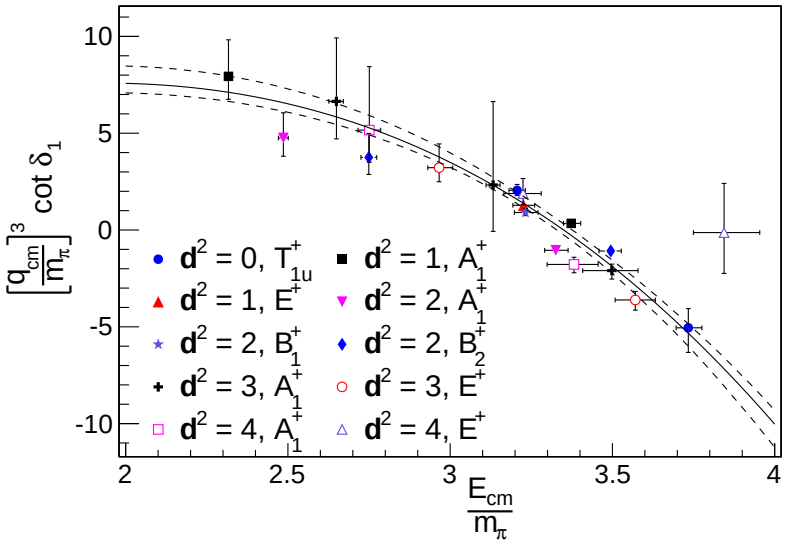
$$(\widetilde{K}^{-1})_{11} = \frac{6\pi E_{\rm cm}}{g^2 m_\pi}\left(\frac{m_\rho^2}{m_\pi^2} - \frac{E_{\rm cm}^2}{m_\pi^2}\right)$$

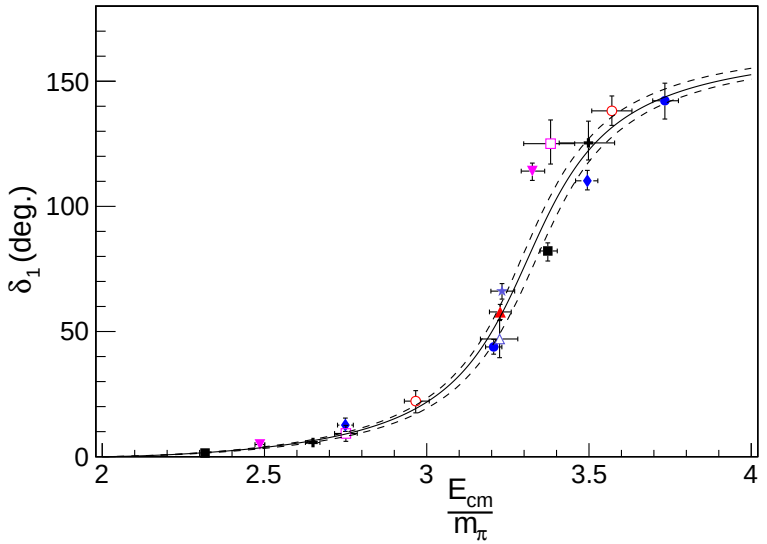
$$(\widetilde{K}^{-1})_{33} = \frac{1}{m_\pi^7 a_3} \qquad (\widetilde{K}^{-1})_{55} = \frac{1}{m_\pi^{11} a_5}$$

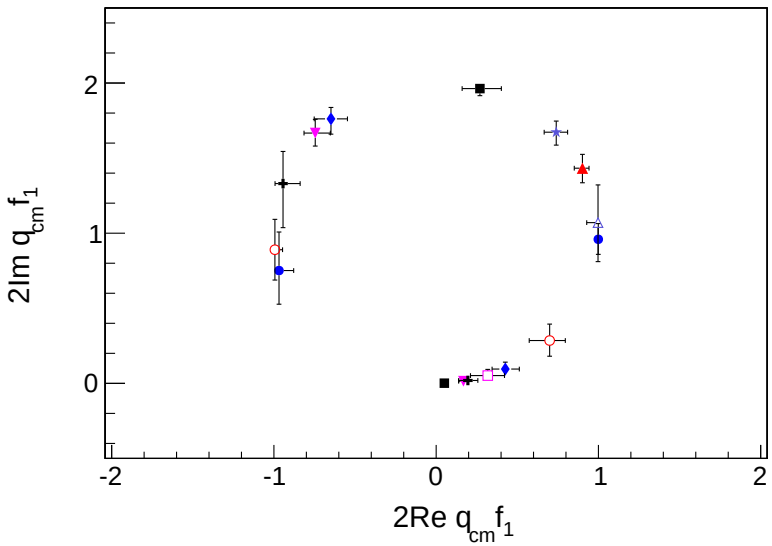
- results

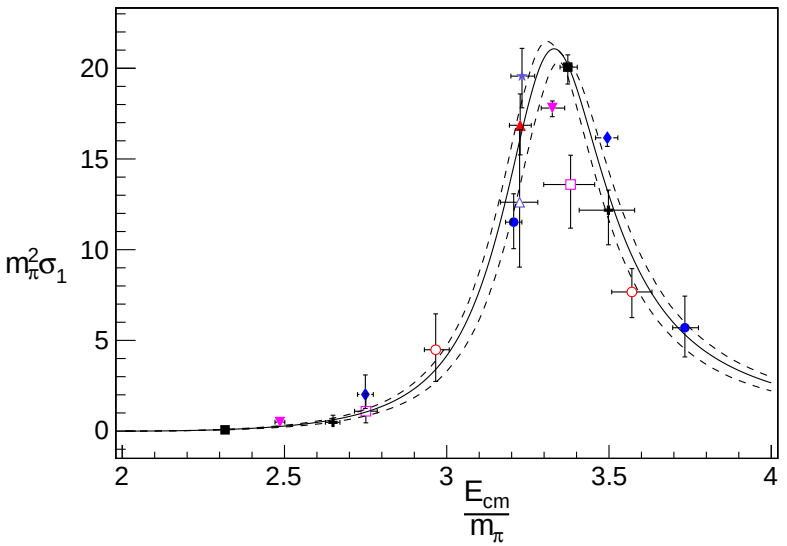
$$\frac{m_\rho}{m_\pi} = 3.349(25),\ g = 5.97(27),\ m_\pi^7 a_3 = -0.00021(100),$$
$$m_\pi^{11} a_5 = -0.00006(24),\ \chi^2/\mathrm{dof} = 1.15$$

Decay of ρ


$\left[\frac{q_{cm}}{m_\pi}\right]^3 \cot \delta_1$
$\frac{E_{cm}}{m_\pi}$
$d^2 = 0, T_{1u}^+$
$d^2 = 1, A_1^+$
$d^2 = 1, E^+$
$d^2 = 2, A_1^+$
$d^2 = 2, B_1^+$
$d^2 = 2, B_2^+$
$d^2 = 3, A_1^+$
$d^2 = 3, E^+$
$d^2 = 4, A_1^+$
$d^2 = 4, E^+$

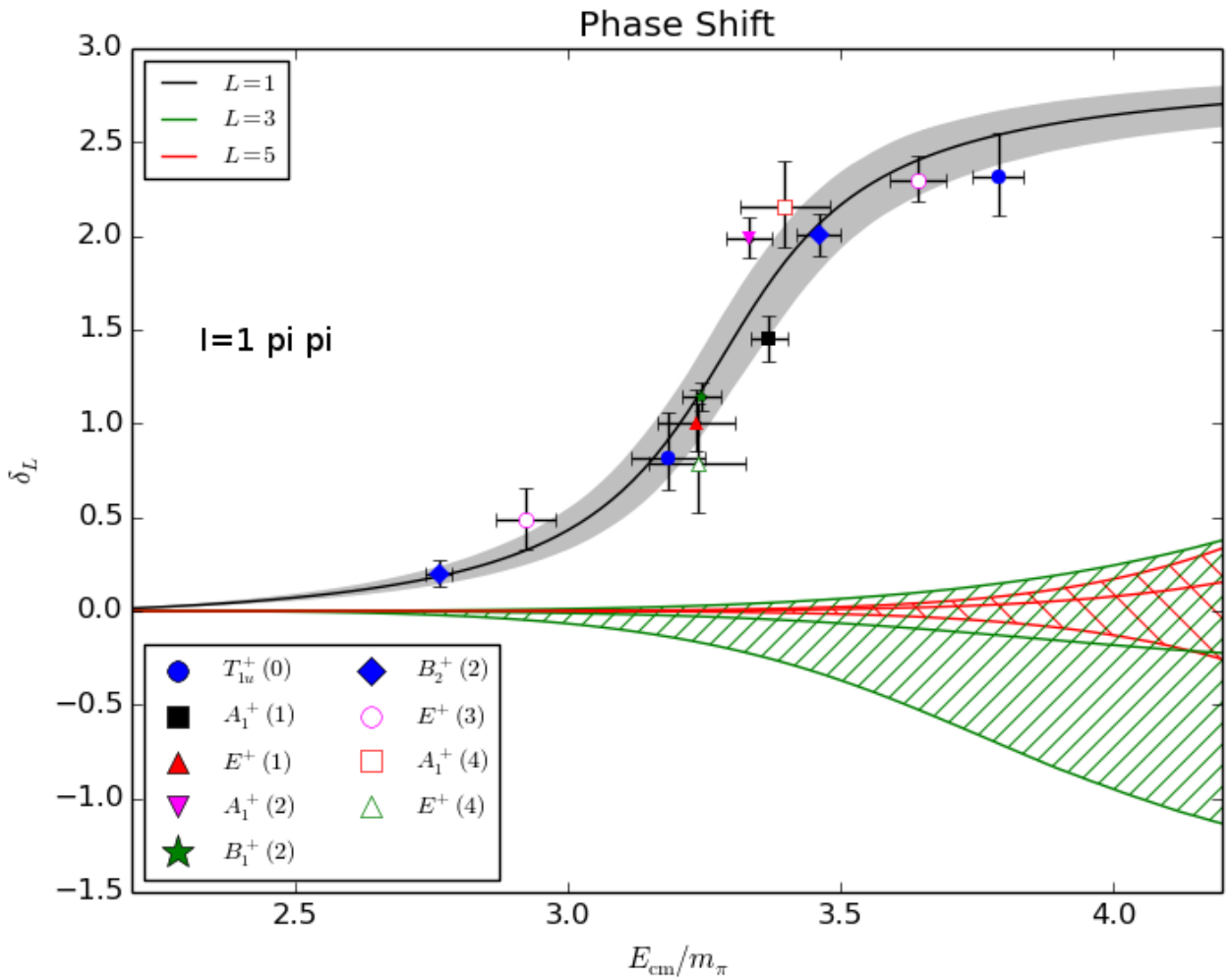

δ_1 (deg.)
$\frac{E_{cm}}{m_\pi}$


$2\mathrm{Im}\, q_{cm} f_1$
$2\mathrm{Re}\, q_{cm} f_1$


$m_\pi^2 \sigma_1$
$\frac{E_{cm}}{m_\pi}$

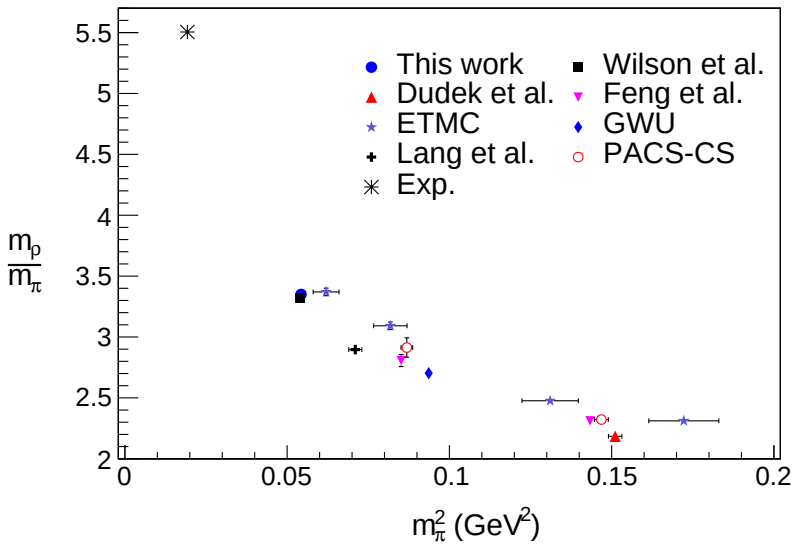
Decay of ρ

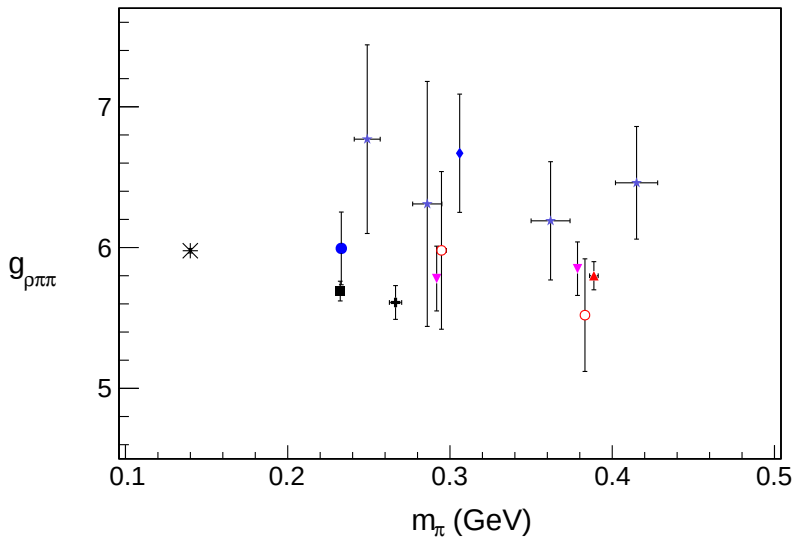
- plot of phase shifts



Comparison to other works

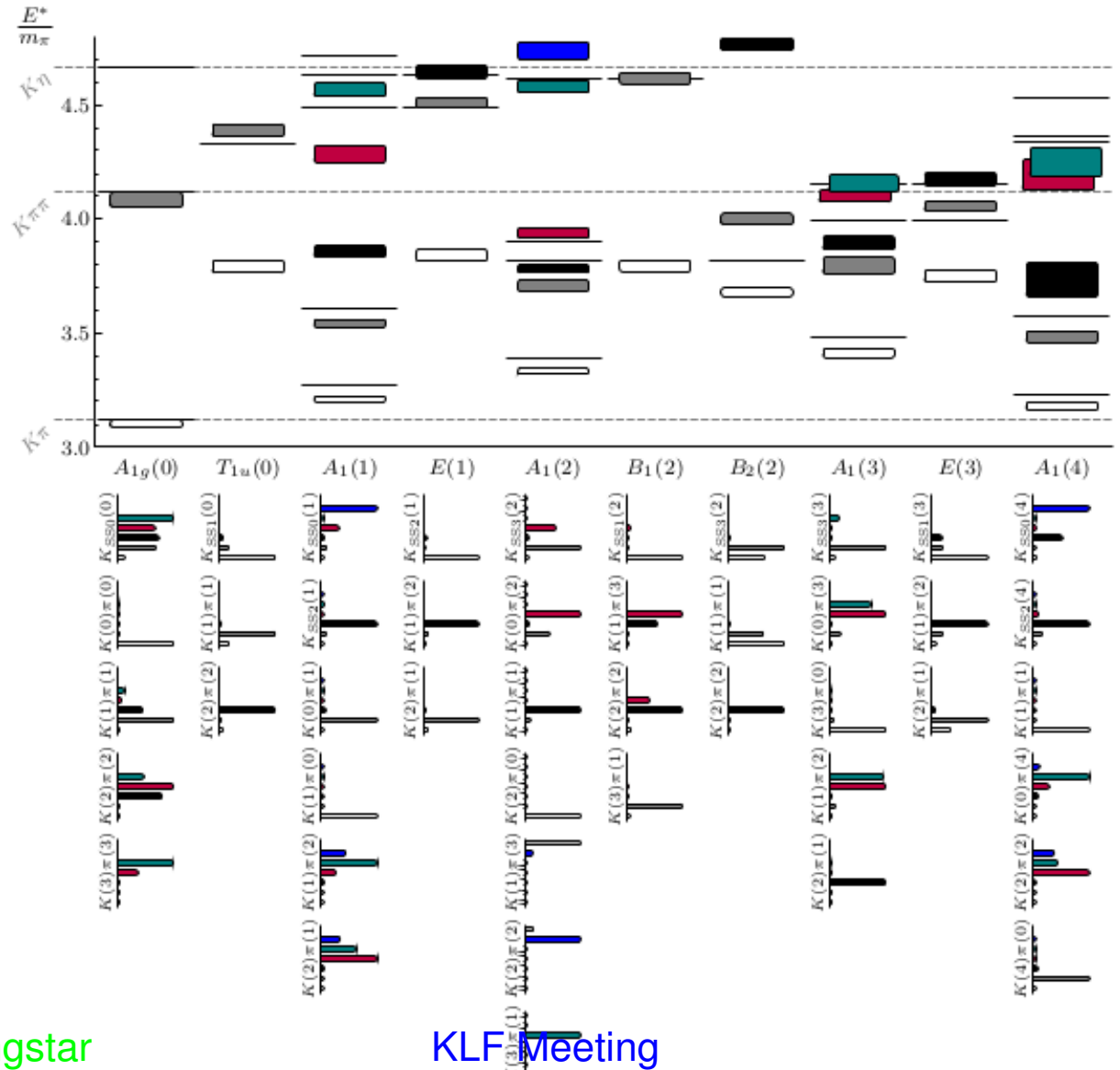
- comparison of our m_ρ and $g_{\rho\pi\pi}$ to other works





$K\pi$ energies in finite volume

- finite volume energies $32^3 \times 256$ lattice, $m_\pi \approx 240$ MeV



Decay of $K^*(892)$

- studied $K^*(892)$
- included $L = 0, 1, 2$ partial waves
- large $32^3 \times 256$ anisotropic lattice, $m_\pi \approx 240$ MeV
- fit forms

$$(\widetilde{K}^{-1})_{11} = \frac{6\pi E_{\rm cm}}{g^2_{K^*\pi\pi} m_\pi}\left(\frac{m^2_{K^*}}{m^2_\pi} - \frac{E^2_{\rm cm}}{m^2_\pi}\right) \qquad (\widetilde{K}^{-1})_{22} = \frac{-1}{m^5_\pi a_2}$$

- S-wave forms tried:

$$\begin{aligned}
(\widetilde{K}^{-1})^{\rm lin}_{00} &= a_{\rm l} + b_{\rm l} E_{\rm cm}, \\
(\widetilde{K}^{-1})^{\rm quad}_{00} &= a_{\rm q} + b_{\rm q} E^2_{\rm cm}, \\
(\tilde{K}^{-1})^{\rm ERE}_{00} &= \frac{-1}{m_\pi a_0} + \frac{m_\pi r_0}{2}\frac{q^2_{\rm cm}}{m^2_\pi}, \\
(\tilde{K}^{-1})^{\rm BW}_{00} &= \left(\frac{m^2_{K^*_0}}{m^2_\pi} - \frac{E^2_{\rm cm}}{m^2_\pi}\right)\frac{6\pi m_\pi E_{\rm cm}}{g^2_{K^*_0\pi\pi} m^2_{K^*_0}}
\end{aligned}$$

K-matrix fits

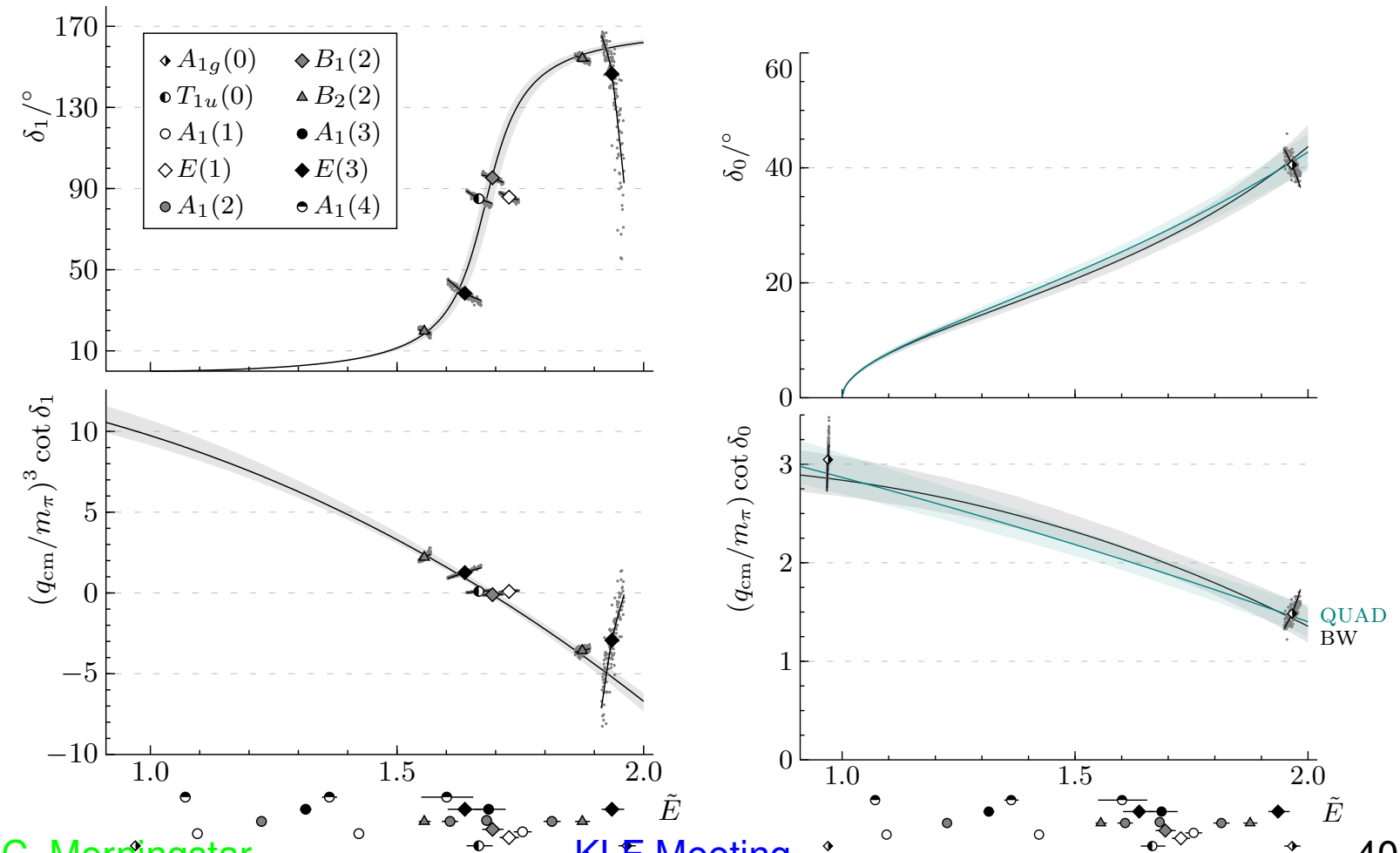
- summary of fit results

Fit	s-wave par.	m_{K^*}/m_π	$g_{K^*K\pi}$	$m_\pi a_0$	χ^2/d.o.f.
(1a,1b)	LIN	$3.819(20)$	$5.54(25)$	$-0.333(31)$	(1.04,–)
2	LIN	$3.810(18)$	$5.30(19)$	$-0.349(25)$	1.49
3	QUAD	$3.810(18)$	$5.31(19)$	$-0.350(25)$	1.47
4	ERE	$3.809(17)$	$5.31(20)$	$-0.351(24)$	1.47
5	BW	$3.808(18)$	$5.33(20)$	$-0.353(25)$	1.42
6	BW	$3.810(17)$	$5.33(20)$	$-0.354(25)$	1.50

- $q\bar{q}$ operators in $A_{1g}(0)$ channel overlap many eigenvectors
- better energy resolution needed for $K_0^*(800)$ determination (future work)
- from NLO effective range parametrization find $m_R/m_\pi = 4.66(13) - 0.87(18)i$ (consistent with BW fit)

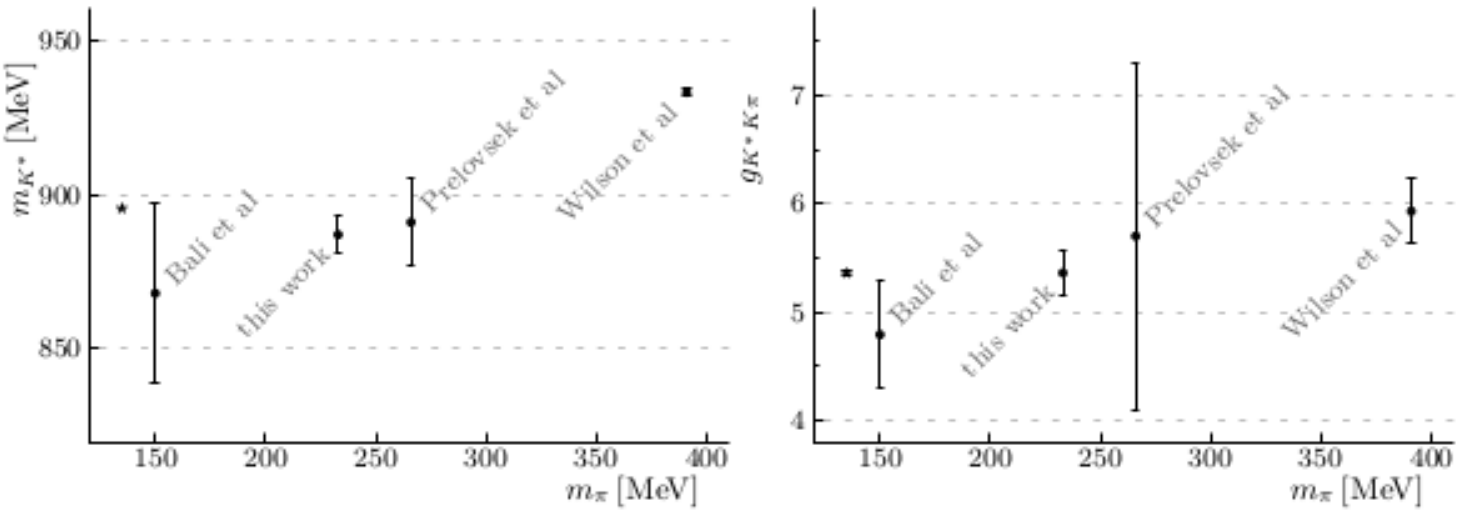
Decay of $K^*(892)$

- plot of P-wave and S-wave phase shift
- included $L = 0, 1, 2$ partial waves
- large $32^3 \times 256$ anisotropic lattice, $m_\pi \approx 240$ MeV
- κ fit: quadratic



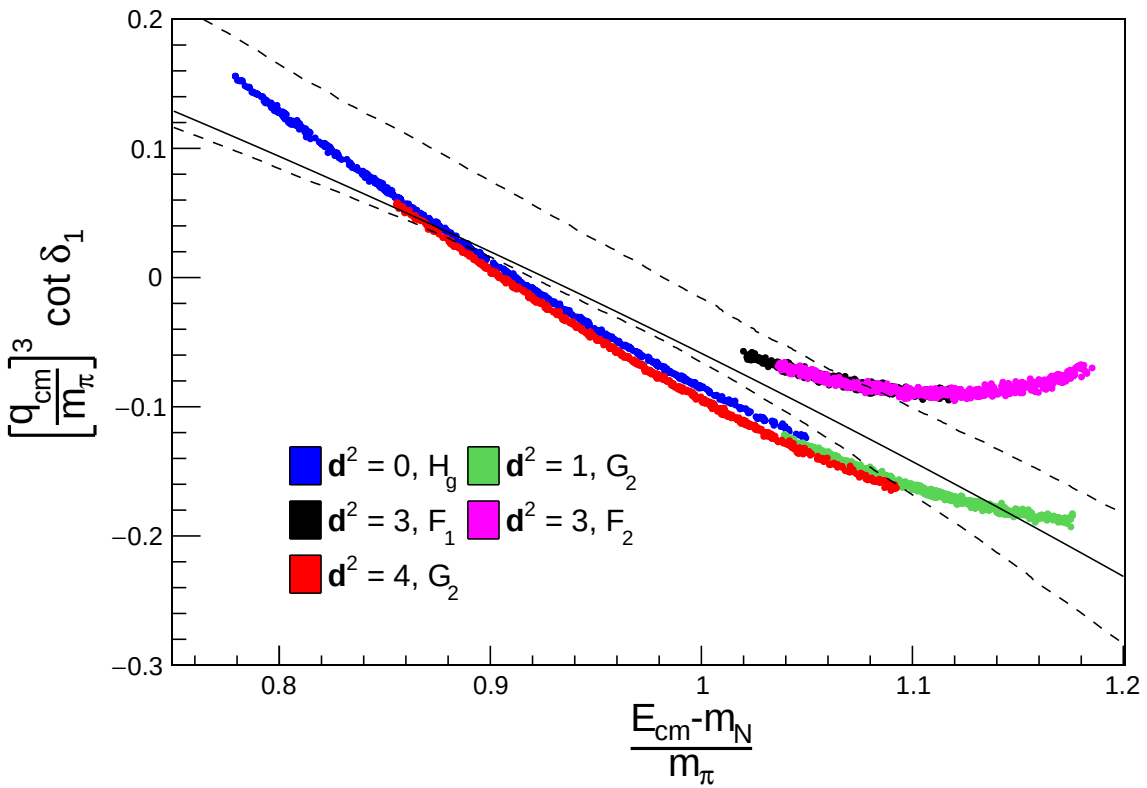
Comparison to other works

- comparison of our m_{K^*} and $g_{K^*K\pi}$ to other works



Decay of Δ

- first baryon resonance decay study
- included $L = 1$ wave only (for now) PRD **97**, 014506 (2018)
- large $48^3 \times 128$ isotropic lattice, $m_\pi \approx 280$ MeV, $a \sim 0.076$ fm
- Breit-Wigner fit gives $g_{\Delta N\pi} = 19.0(4.7)$ in agreement with experiment ~ 16.9



Recent $I = 1$ $\pi\pi$ scattering results

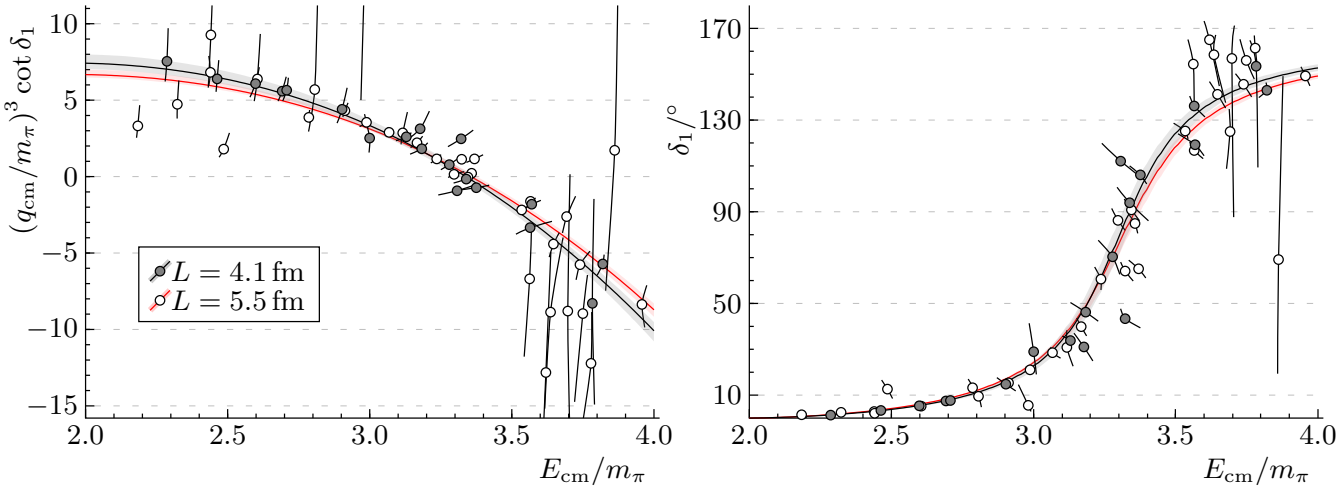
- Most recent work on CLS ensembles NPB**939**, 145 (2019)

ID	$a\,[\mathrm{fm}]$	$L^3 \times T$	$m_\pi,\, m_K\,[\mathrm{MeV}]$	N_{conf}
C101	0.086	$48^3 \times 96$	220, 470	300
D101		$64^3 \times 128$		303
N401	0.076	$48^3 \times 128$	280, 460	274
N200	0.064	$48^3 \times 128$	280, 460	854
D200		$64^3 \times 128$	200, 480	558
J303	0.050	$64^3 \times 192$	260, 470	328

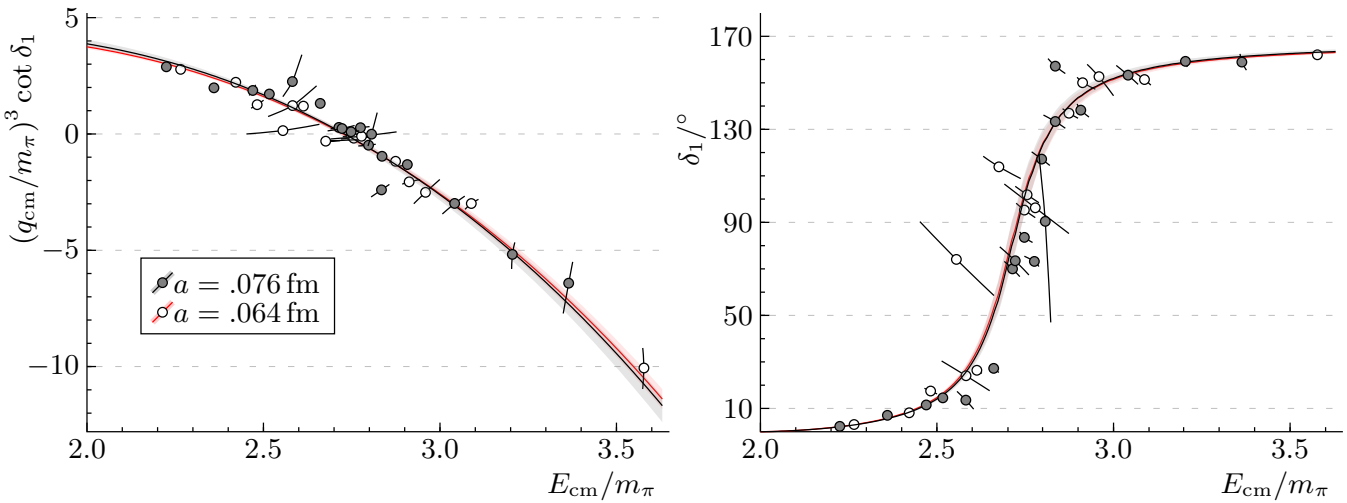
ID	(ρ, n_ρ)	N_{ev}	dilution	N_{t_0}	N_{inv}
C101	$(0.1, 20)$	392	$(\mathrm{TF}, \mathrm{SF}, \mathrm{LI16})_\mathrm{F}\,(\mathrm{TI8}, \mathrm{SF}, \mathrm{LI16})_\mathrm{R}$	1	1408
D101		928	$(\mathrm{TF}, \mathrm{SF}, \mathrm{LI16})_\mathrm{F}\,(\mathrm{TI8}, \mathrm{SF}, \mathrm{LI16})_\mathrm{R}$	2	1792
N401	$(0.1, 25)$	320	$(\mathrm{TF}, \mathrm{SF}, \mathrm{LI16})_\mathrm{F}\,(\mathrm{TI8}, \mathrm{SF}, \mathrm{LI16})_\mathrm{R}$	2	1664
N200	$(0.1, 36)$	192	$(\mathrm{TF}, \mathrm{SF}, \mathrm{LI8})_\mathrm{F}\,(\mathrm{TI8}, \mathrm{SF}, \mathrm{LI8})_\mathrm{R}$	2	832
D200		448	$(\mathrm{TF}, \mathrm{SF}, \mathrm{LI8})_\mathrm{F}\,(\mathrm{TI8}, \mathrm{SF}, \mathrm{LI8})_\mathrm{R}$	2	832
J303	$(0.1, 60)$	208	$(\mathrm{TF}, \mathrm{SF}, \mathrm{LI8})_\mathrm{F}\,(\mathrm{TI16}, \mathrm{SF}, \mathrm{LI8})_\mathrm{R}$	3	1504

Decay of the ρ

- scattering phase shift for C101 and D101

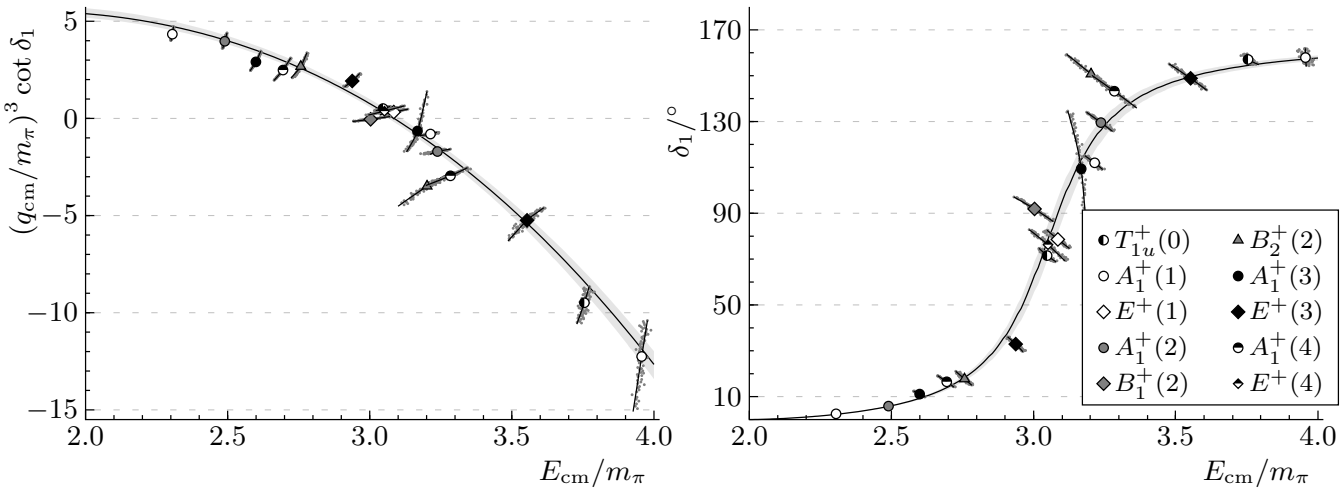


- scattering phase shift for N200 and N401

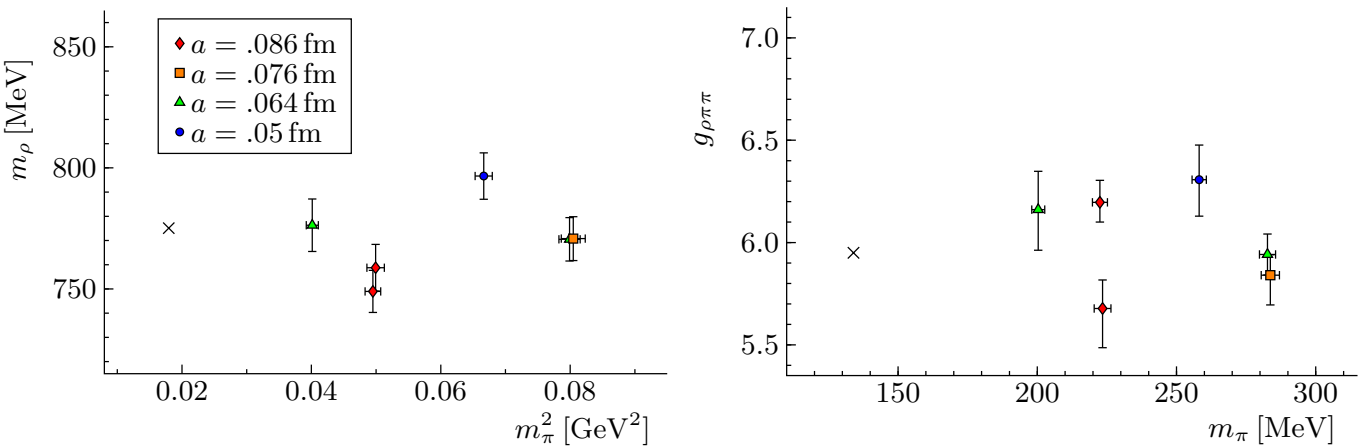


Decay of the ρ

- scattering phase shift for J303



- summary of resonance parameters



Time-like pion form factor

- obtain time-like pion form factor NPB**939**, 145 (2019) from

$$|F_\pi(E_{\rm cm})|^2 = g_\Lambda(\gamma)\left(q_{\rm cm}\frac{\partial\delta_1}{\partial q_{\rm cm}} + u\frac{\partial\phi_1^{(\boldsymbol{d},\Lambda)}}{\partial u}\right)\frac{3\pi E_{\rm cm}^2}{2q_{\rm cm}^5 L^3}|\langle 0|V^{(\boldsymbol{d},\Lambda)}|\boldsymbol{d}\Lambda n\rangle|^2$$

  where

$$\gamma = \frac{E}{E_{\rm cm}}, \quad u = \frac{Lq_{\rm cm}}{2\pi}, \quad g_\Lambda(\gamma) = \begin{cases} \gamma^{-1}, & \Lambda = A_1^+ \\ \gamma, & \text{otherwise} \end{cases}$$

  and δ_1 is the physical phase shift, and $B_{11}^{(\boldsymbol{d},\Lambda)} = (q_{\rm cm}/m_\pi)^3\cot\phi_1^{(\boldsymbol{d},\Lambda)}$ gives the pseudophase $\phi_1^{(\boldsymbol{d},\Lambda)}$

- we compute the matrix element

$$V^{(\boldsymbol{d},\Lambda)} = \sum_\mu b_\mu^{(\boldsymbol{d},\Lambda)} V_{R,\mu}, \qquad \sum_\mu b_\mu^{(\boldsymbol{d},\Lambda)*} b_\mu^{(\boldsymbol{d},\Lambda)} = 1,$$

$$V_{R,\mu} = Z_V(1+ab_V m_1 + a\bar{b}_V \mathrm{Tr}M_q)\, V_{I,\mu}, \qquad V_{I,\mu} = V_\mu + ac_V\widetilde{\partial}_\nu T_{\mu\nu},$$

$$V_\mu^a = \tfrac{1}{2}\overline{\psi}\gamma_\mu\tau^a\psi, \qquad \widetilde{\partial}_\nu T_{\mu\nu}^a = \tfrac{1}{2}i\widetilde{\partial}_\nu\overline{\psi}\sigma_{\mu\nu}\tau^a\psi$$

Time-like pion form factor (con't)

- renormalization constants determined nonperturbatively
- tried Gounaris-Sakurai parametrization

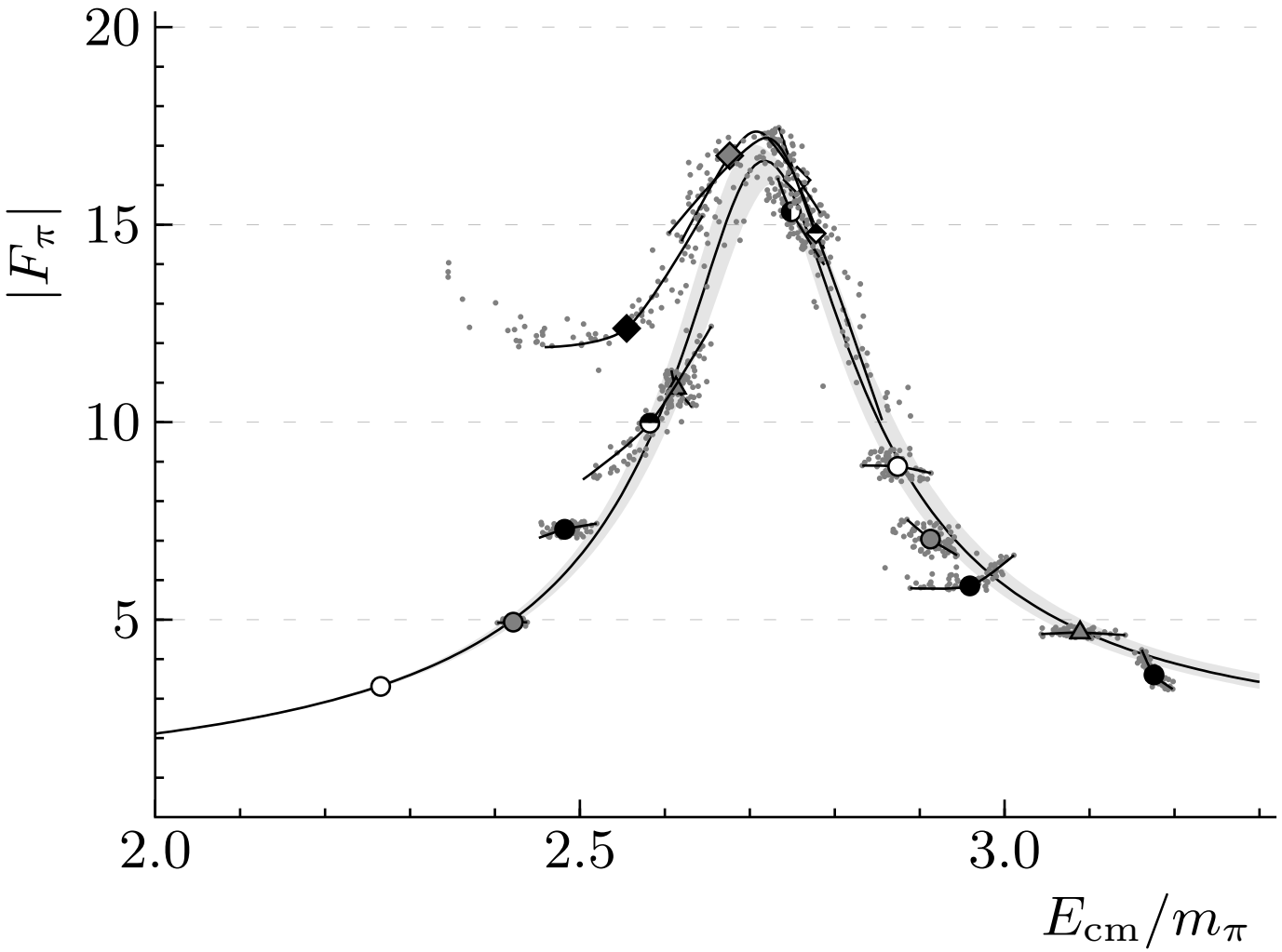
$$
\begin{aligned}
F_\pi^{GS}(\sqrt{s}) &= \frac{-\frac{m_\pi^2}{\pi} - q_\rho^2 h(m_\rho) - b\frac{m_\rho^2}{4}}{q_{\rm cm}^2 h(\sqrt{s}) - q_\rho^2 h(m_\rho) + b(q_{\rm cm}^2 - q_\rho^2) - \frac{q_{\rm cm}^3}{\sqrt{s}} i}, \\
b &= -h(m_\rho) - \frac{24\pi}{g_{\rho\pi\pi}^2} - \frac{2q_\rho^2}{m_\rho} h'(m_\rho), \\
h(\sqrt{s}) &= \frac{2}{\pi}\frac{q_{\rm cm}}{\sqrt{s}} \ln\left(\frac{\sqrt{s} + 2q_{\rm cm}}{2m_\pi}\right),
\end{aligned}
$$

- also tried n-subtracted dispersion relation with Omnes-Muskhelishvili solution

$$
F_\pi(s) = \exp\left(\sum_{k=1}^{n} p_k s^k\right) \exp\left\{\frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty} \frac{dz}{z^n}\frac{\delta_1(z)}{z - s - i\epsilon}\right\}
$$

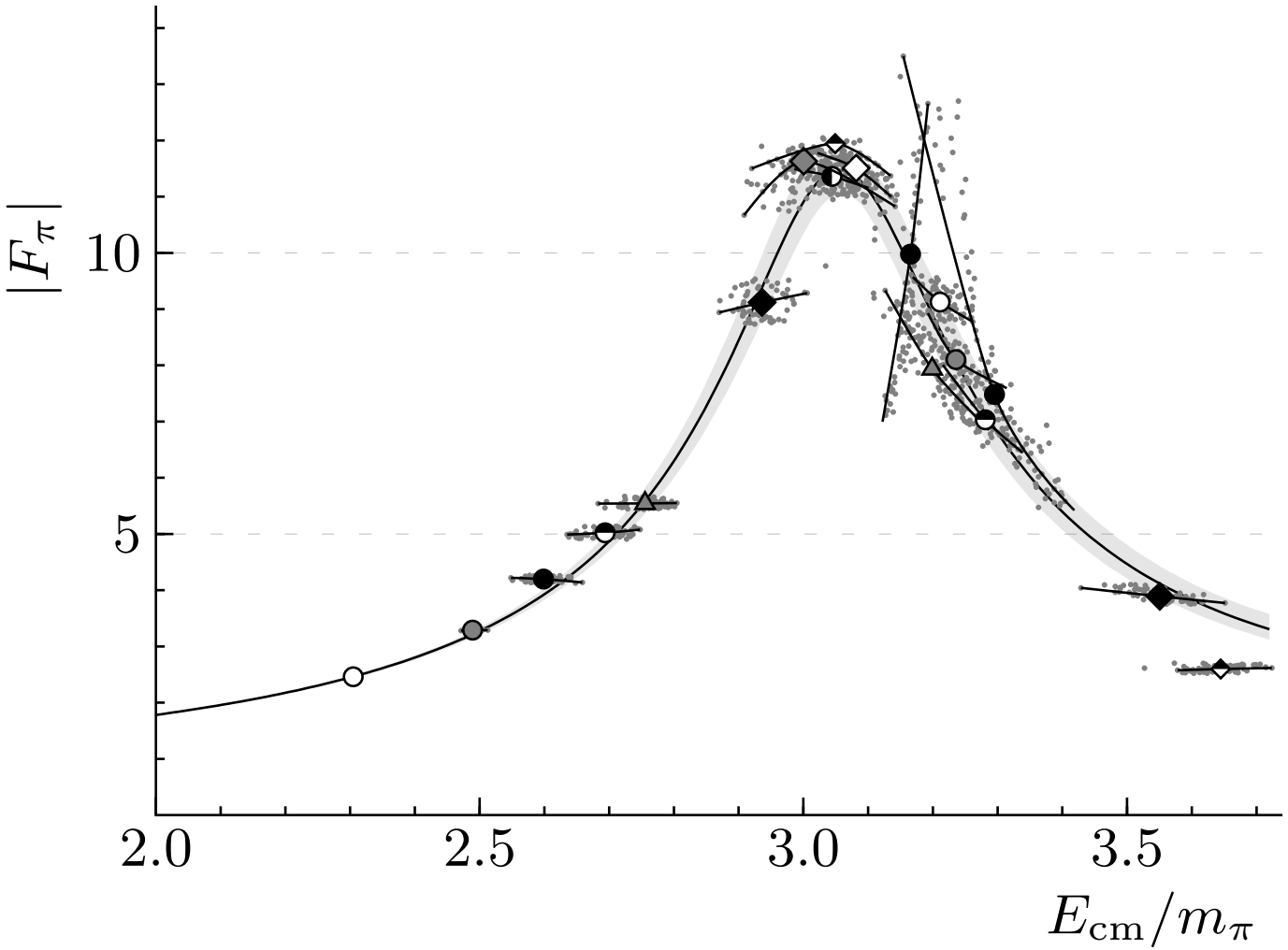
Time-like pion form factor results

- results for N200 (curve is fit with thrice-subtracted dispersion)



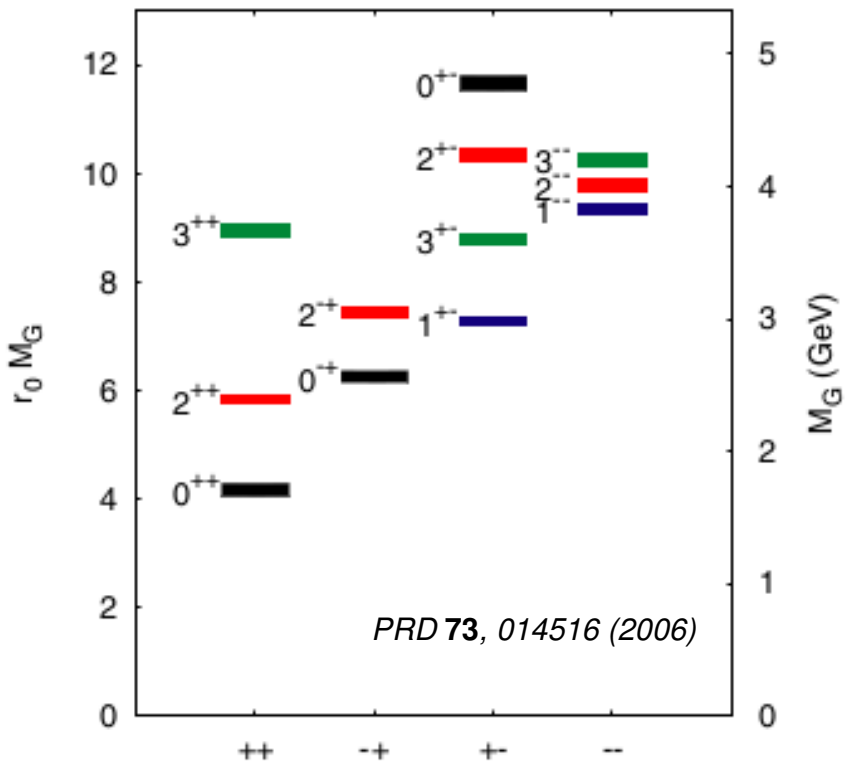
Time-like pion form factor results

- results for J303 (curve is fit with thrice-subtracted dispersion)



The scalar glueball

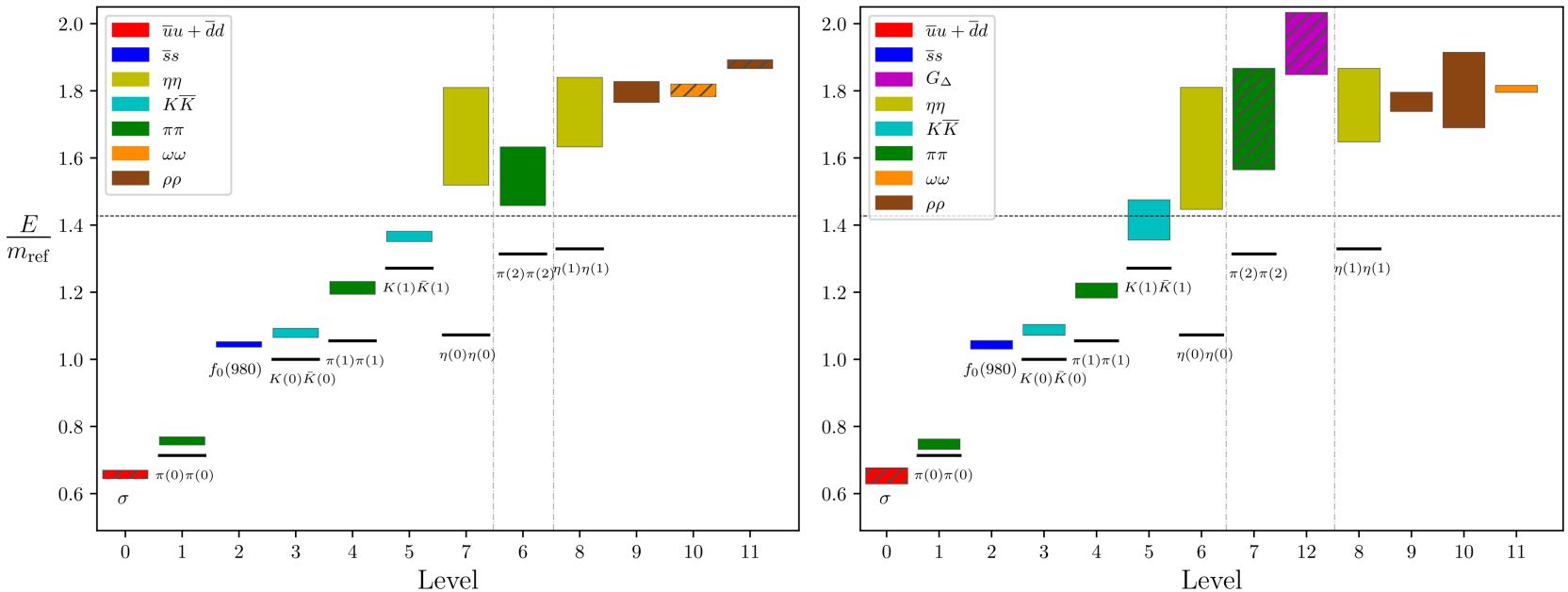
- glueball: hypothetical bound state of gluons
- experimental evidence elusive, light scalar candidates:
  - $f_0(1370), f_0(1500), f_0(1710)$
- lattice studies to date:
  - light scalar $\sim 1600 - 1700$ MeV
  - most in pure $SU(3)$ or quenched approx. (no quark/meson mixing!)



- here: extract low-lying A_{1g}^+ spectrum with $q\bar{q}$, meson-meson, & glueball operators
  - first look (from the lattice) at mixing between glueball, $q\bar{q}$, and two-hadron states

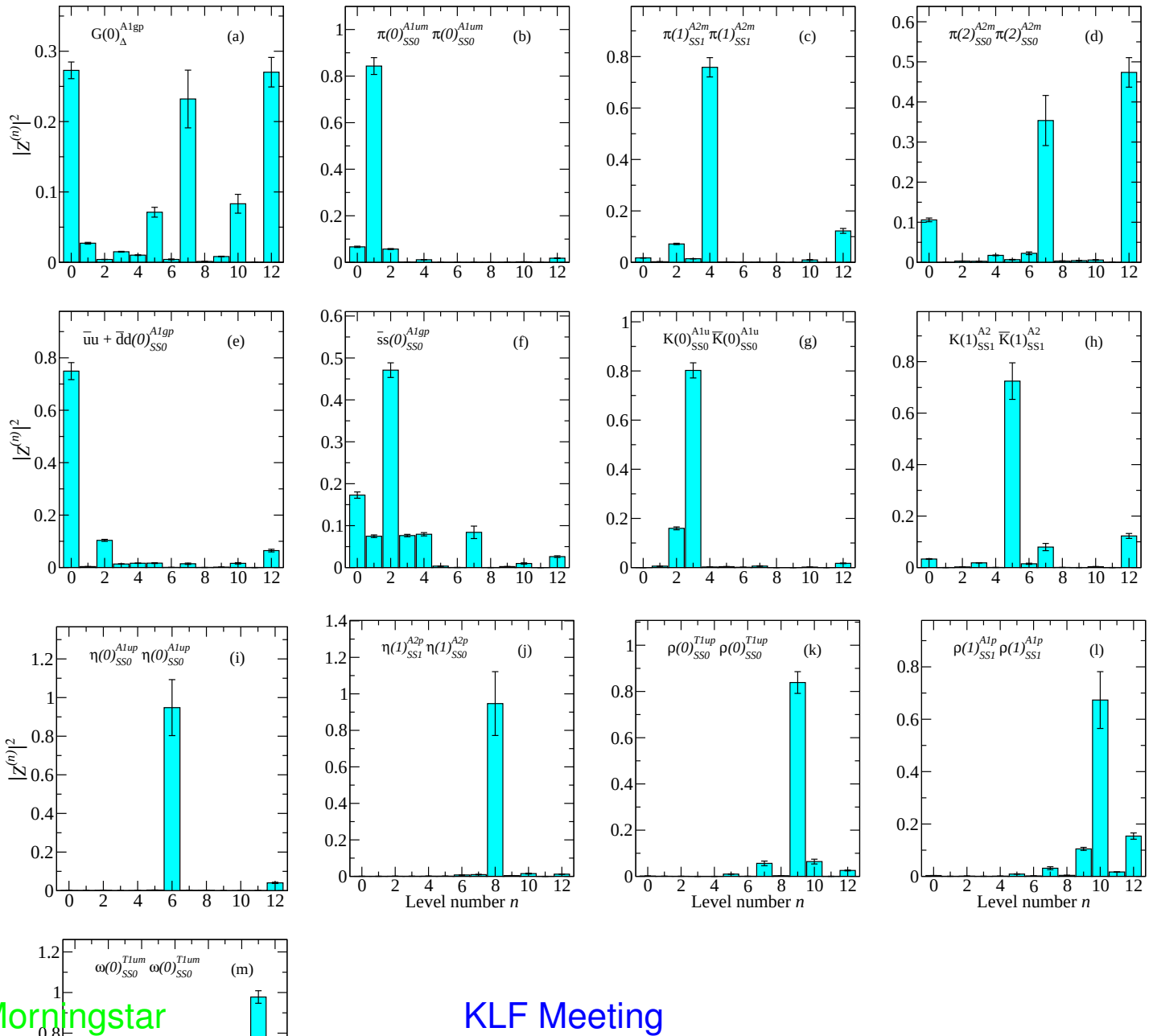
A_{1g}^+ spectrum

- $24^3 \times 128$ anisotropic lattice, $m_\pi \sim 390$ MeV



- bad news for the scalar glueball?

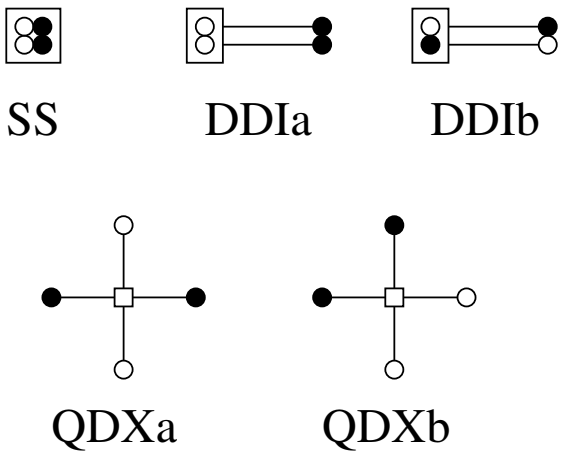
A_{1g}^{+} overlaps



Tetraquark operators

- determine impact on spectrum when tetraquark operators included
- tetraquark operators we study are combinations of

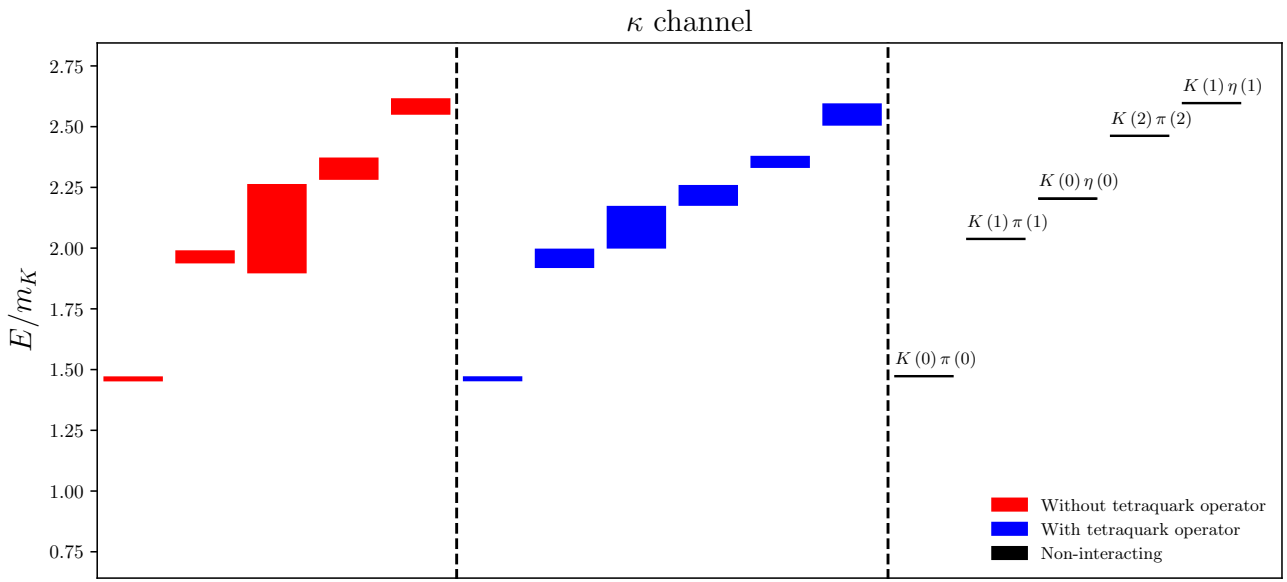
$$\Phi^{ABCD\pm}_{\alpha\beta\mu\nu;ijkl}(t) = \sum_{\boldsymbol{x}} e^{-i\boldsymbol{p}\cdot\mathbf{x}}(\delta_{ab}\delta_{cd} \pm \delta_{ad}\delta_{bc}) \\ \overline{q}^{A}_{a\alpha i}(\boldsymbol{x},t)\; q^{B}_{b\beta j}(\boldsymbol{x},t)\; \overline{q}^{C}_{c\mu k}(\boldsymbol{x},t)\; q^{D}_{d\nu l}(\boldsymbol{x},t),$$



- spectrum in $\kappa,\ a_0(980)$ channels using $32^3 \times 256$ anisotropic lattice $m_\pi \approx 235$ MeV, $N_{\text{cfgs}} = 412$, $\xi \approx 3.451$.

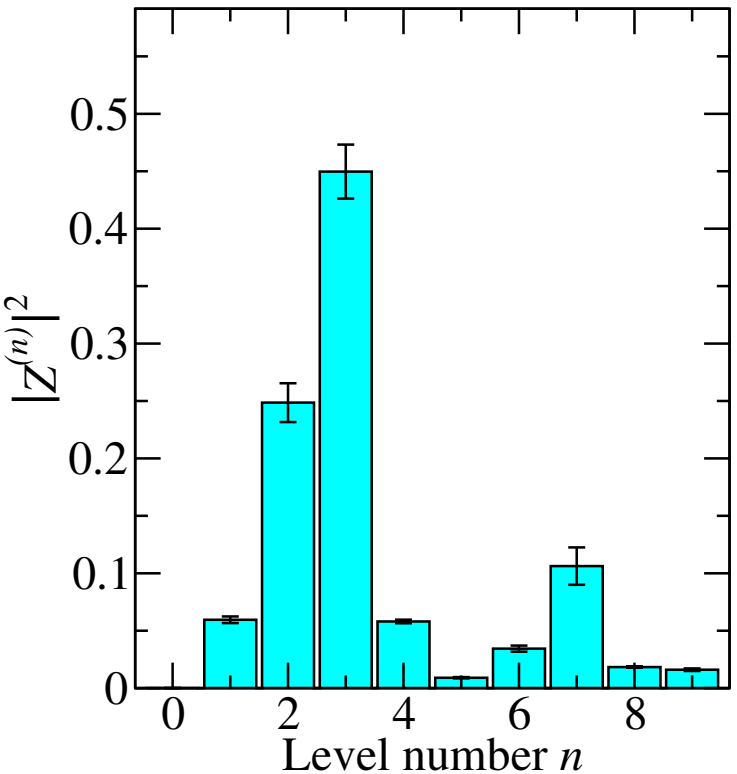
$I = \frac{1}{2},\ S = 1,\ A_{1g}$ channel for $K_0^*(700)$

- single-site and displaced $q\bar{q}$, $K\pi$, $K\phi$, and $K\eta$ operators
- tried several hundred tetraquark operators, different flavor structures
- found a dozen or so operators yield additional energy level
- never got two extra levels



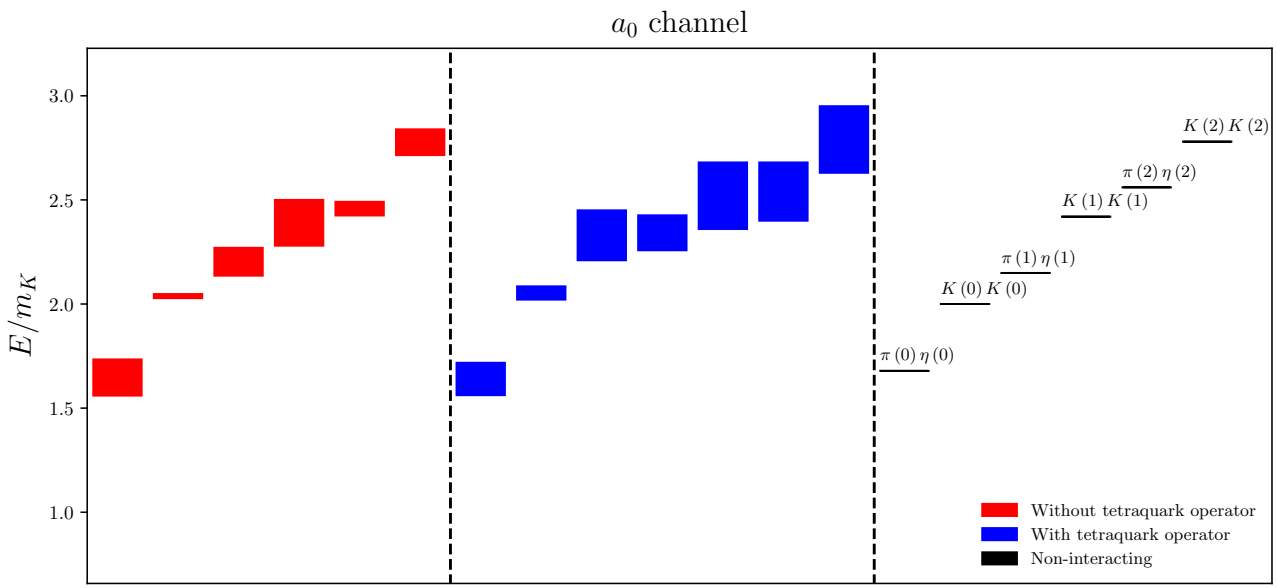
$I = \frac{1}{2},\ S = 1,\ A_{1g}$ channel for $K_0^*(700)$ (con't)

- Overlap factors for single-site $\bar{s}u\bar{s}s$ color antisym tetraquark operator



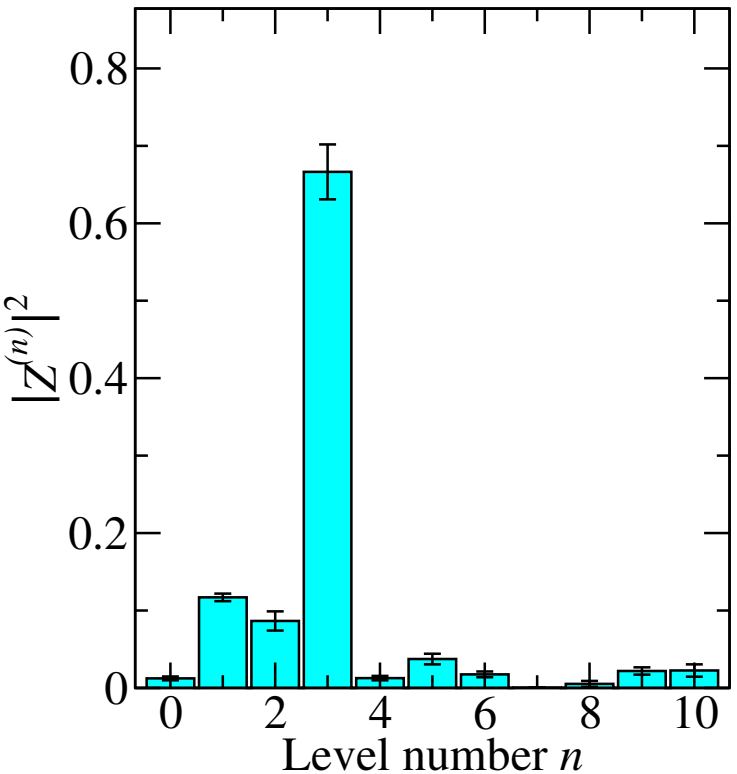
$I = 1,\ S = 0,\ A_{1g}^{-}$ channel for $a_0(980)$

- single-site and displaced $q\bar{q}$, $K\bar{K}$, $\eta\pi$, and $\phi\pi$ operators
- tried several hundred tetraquark operators, different flavor structures
- found a dozen or so operators yield additional energy level
- never got two extra levels



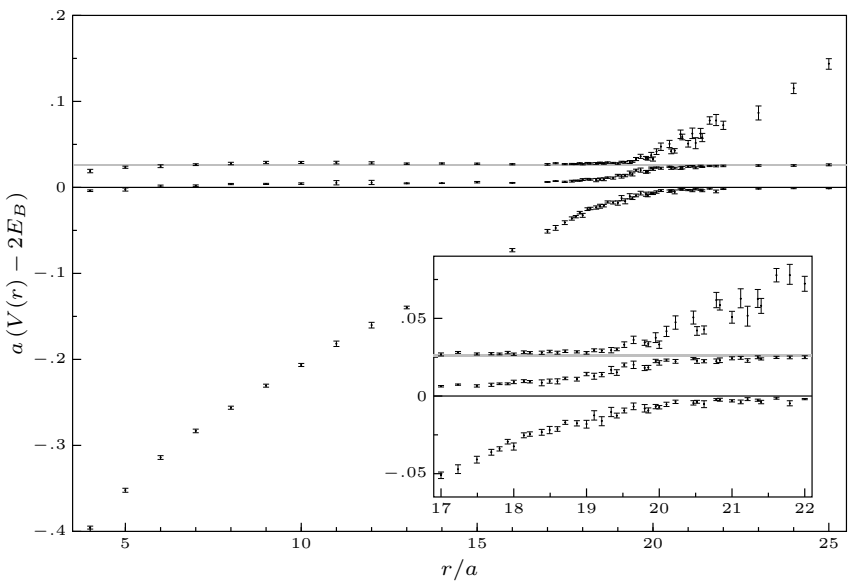
$I = 1,\ S = 0,\ A_{1g}^{-}$ channel for $a_0(980)$ (con't)

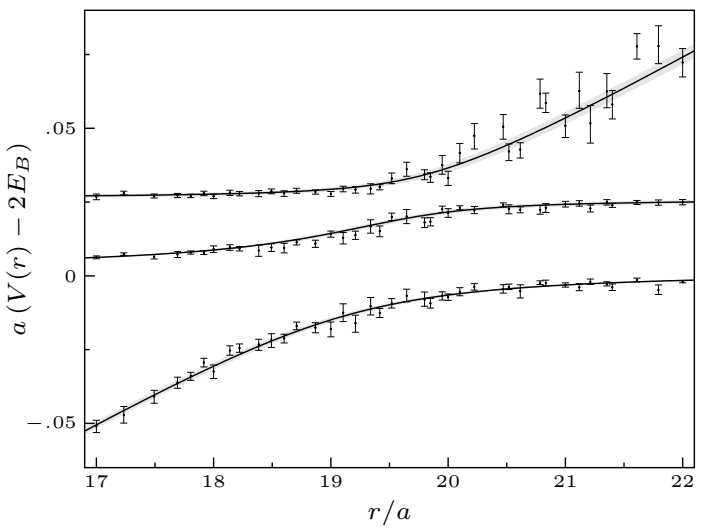
- Overlap factors for single-site $\overline{u}u\overline{d}s$ color antisym tetraquark operator



String Breaking

- string breaking (with Vanessa Koch et al.) PLB 793, 493 (2019)
- Wilson line operator with static-light $B\overline{B}$ and $B_s\overline{B}_s$ operators
- use of stochastic LapH method
- $N_f = 2+1$ CLS N200 ensemble $48^3 \times 128,\ a = 0.064$ fm, $m_\pi 280$ MeV, $m_K = 460$ MeV





Collaborators

- people involved in this work:

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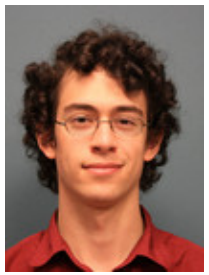

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- thanks to NSF XSEDE:
  - Stampede at TACC
  - Comet at SDSC

Current Efforts

- recently joined with CalLat (Walker-Loud and others) and U. Mainz (Wittig and others) to study baryon-baryon scattering
- major software redevelopment: common subexpression elimination with tensor contractions, heavy use of batched BLAS routines (with Ben Hörz)
- DOE INCITE award on Summit at ORNL
- large volumes near physical point
- $NN, N\Sigma, N\Lambda, \Lambda\Lambda, N\Xi$ included
- also includes $N\overline{K}, N\pi, \Sigma\pi$
- $\Lambda(1405), \Delta(1232)$, H-dibaryon, and Roper to be studied
- Δ transition form factors

Proton mass decomposition

- recent determination of mass decomposition of proton
  [Y.Yang, J.Liang, Y.Bi, Y.Chen, T.Draper, K.F.Liu, Z.Liu, PRL121, 212001 (2018)]
- rest mass M of proton given by [Ji PRL74, 1071 (1995)]

$$M = -\langle T_{44} \rangle = \langle H_m \rangle + \langle H_E \rangle(\mu) + \langle H_g \rangle(\mu) + \tfrac{1}{4}\langle H_a \rangle,$$

- $\langle T_{\mu\nu} \rangle$ expectation value of energy momentum tensor in hadron
- quark condensate $H_m = \sum_{u,d,s\ldots} \int d^3x \, m \, \overline{\psi}\psi$
- quark energy $H_E = \sum_{u,d,s\ldots} \int d^3x \, \overline{\psi}(\vec{D} \cdot \vec{\gamma})\psi$
- glue field energy $H_g = \int d^3x \, \frac{1}{2}(B^2 - E^2)$
- anomaly term
  $H_a = \sum_{u,d,s\ldots} \int d^3x \, \gamma_m m \, \overline{\psi}\psi - \int d^3x \, \frac{\beta(g)}{g}(E^2 + B^2)$
- $\langle H_m \rangle$, $\langle H_a \rangle$, $\langle H_E + H_g \rangle$ scale and scheme independent
- obtain from renormalized quark and gluon momentum fractions
  $\langle H_g \rangle = \frac{3}{4} M \langle x \rangle_g$ and $\langle H_E \rangle = \frac{3}{4} M \langle x \rangle_q - \frac{3}{4}\langle H_m \rangle$
- anomaly term from $\langle H_a \rangle = M - \langle H_m \rangle$

Proton mass decomposition (con't)

- determined mass M from two-point correlator
- used previous determination of $\langle H_m \rangle$ (2016)
- momentum fractions from

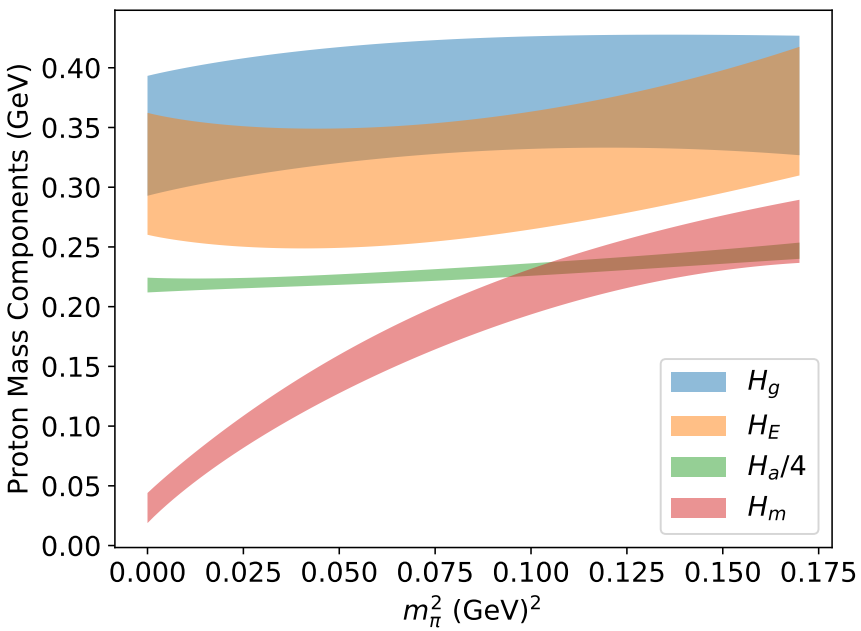
$$\begin{aligned}
\langle x \rangle_{q,g} &\equiv -\frac{\langle N | \frac{4}{3} \overline{T}^{q,g}_{44} | N \rangle}{M \langle N | N \rangle}, \\
\overline{T}^{q}_{44} &= \int d^3x \overline{\psi}(x) \frac{1}{2} (\gamma_4 \overleftrightarrow{D}_4 - \frac{1}{4} \sum_{i=0,1,2,3} \gamma_i \overleftrightarrow{D}_i) \psi(x), \\
\overline{T}^{g}_{44} &= \int d^3x \frac{1}{2} (E(x)^2 - B(x)^2).
\end{aligned}$$

- renormalization

$$\begin{aligned}
\langle x \rangle^R_{u,d,s} &= Z^{\overline{\rm MS}}_{QQ}(\mu) \langle x \rangle_{u,d,s} + \delta Z^{\overline{\rm MS}}_{QQ}(\mu) \sum_{q=u,d,s} \langle x \rangle_q + Z^{\overline{\rm MS}}_{QG}(\mu) \langle x \rangle_g \\
\langle x \rangle^R_g &= Z^{\overline{\rm MS}}_{GQ}(\mu) \sum_{q=u,d,s} \langle x \rangle_q + Z^{\overline{\rm MS}}_{GG} \langle x \rangle_g,
\end{aligned}$$

Proton mass decomposition (con't)

- obtained results on 4 ensembles ($N_f = 2 + 1$ DWF action, overlap valence)
- disconnected insertions: cluster-decomposition error reduction, all time slices looped over
- extrapolate with global fit including finite volume, spacing corrections, chiral behavior



- quark energy $32(4)(4)\%$
- glue energy $36(5)(4)\%$
- quark condensate $9(2)(1)\%$
- trace anomaly $23(1)(1)\%$
- with $N_f = 2 + 1$

Nucleon spin decomposition

- spin decomposition of nucleon
  [C.Alexandrou, M.Constantinou, K.Hadjiyiannakou, K.Jansen, C.Kallidonis, G.Koutsou, A.V.Avilés-Casco, C.Wiese, PRL 119, 142002 (2017)]
- from Ji sum rule [Ji, PRL78, 610 (1997)]

$$J_N = \sum_{q=u,d,s,c\cdots} \left(\tfrac{1}{2}\Delta\Sigma_q + L_q \right) + J_g$$

- obtain from nucleon matrix elements $(Q = p' - p, \quad P = \tfrac{1}{2}(p' + p))$

$$\langle N(p,s')|\bar{q}\gamma_\mu\gamma_5 q|N(p,s)\rangle = \bar{u}_N(p,s')\left[g_A^q \gamma^\mu\gamma_5\right]u_N(p,s),$$

$$\langle N(p',s')|\bar{q}\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}q|N(p,s)\rangle = \bar{u}_N(p',s')\Lambda^q_{\mu\nu}(Q^2)u_N(p,s),$$

$$\Lambda^{\mu\nu}_{q(g)}(Q^2) = A^{q(g)}_{20}(Q^2)\gamma^{\{\mu}P^{\nu\}} + B^{q(g)}_{20}(Q^2)\frac{\sigma^{\{\mu\alpha}q_\alpha P^{\nu\}}}{2m} + C^{q(g)}_{20}(Q^2)\,\frac{1}{m}Q^{\{\mu}Q^{\nu\}},$$

Nucleon spin decomposition (con't)

- quark(gluon) total angular momentum and quark momentum fraction and spin from

$$\begin{aligned} J_{q(g)} &= \tfrac{1}{2}[A_{20}^{q(g)}(0) + B_{20}^{q(g)}(0)] \\ \langle x \rangle_q &= A_{20}^{q}(0), \qquad \Delta\Sigma_q = g_A^q \end{aligned}$$

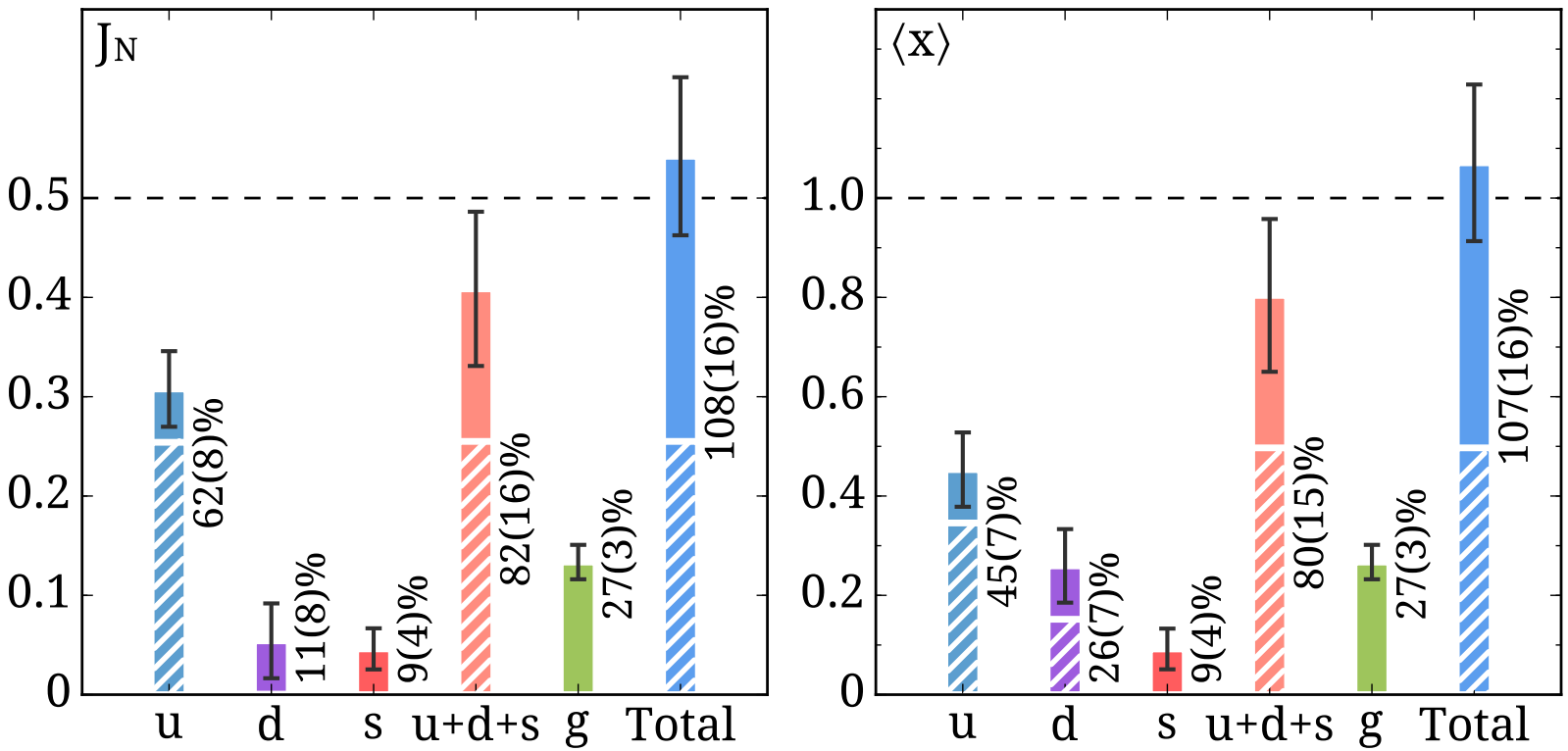
- gluon momentum fraction from $\mathcal{O}^g_{\mu\nu}{=}2\mathrm{Tr}[G_{\mu\sigma}G_{\nu\sigma}]$ with $\overline{\mathcal{O}}^g{\equiv}\mathcal{O}^g_{44} - \frac{1}{3}\mathcal{O}^g_{jj}$

$$\langle N(p,s')|\overline{\mathcal{O}}^g|N(p,s)\rangle{=}\left(-4E_N^2 - \frac{2}{3}\vec{p}^2\right)\langle x\rangle_g,$$

- one ensemble at physical point $48^3 \times 96$ twisted mass clover-improved $a = 0.0939(3)$ fm from nucleon mass
- u, d disconnected diagrams by exact deflation + one-end-trick
- s disconnected diagrams by truncated solver method
- renormalization factors determined nonperturbatively

Nucleon spin decomposition (con't)

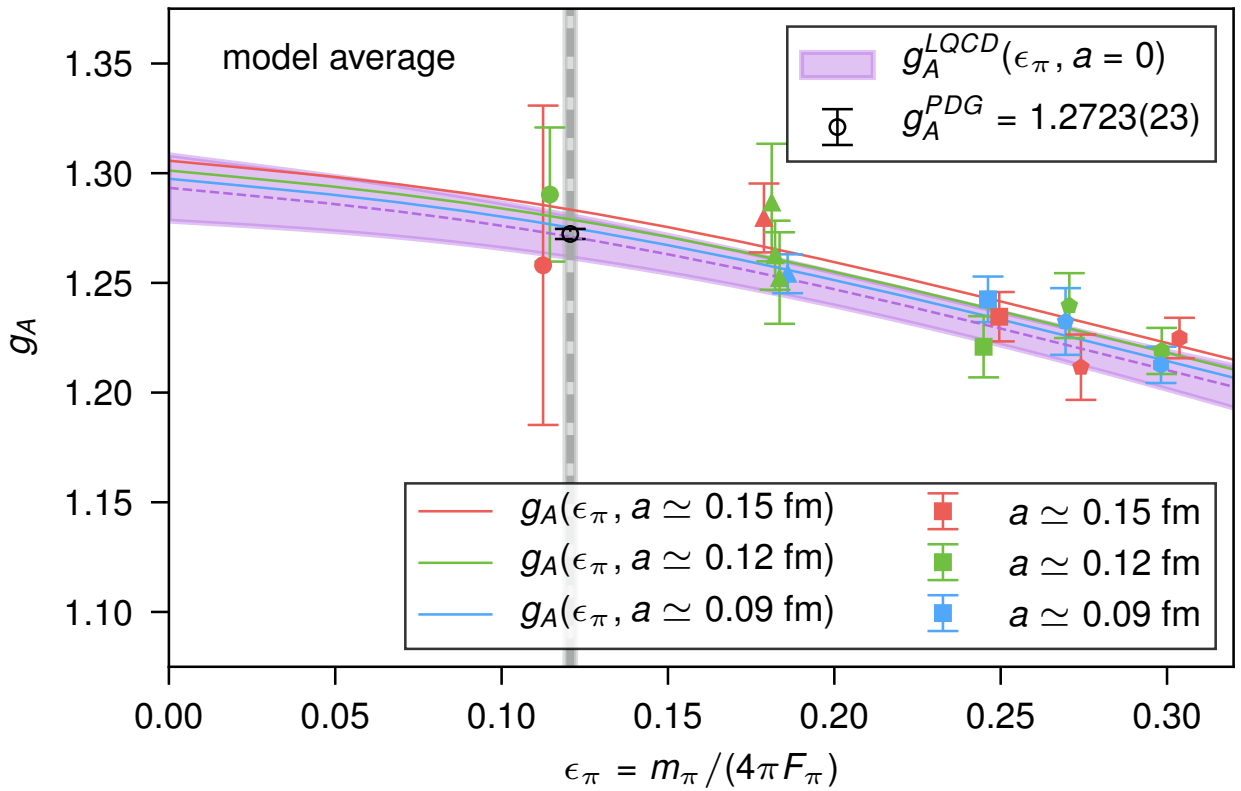
- nucleon spin (left) and momentum (right) decompositions
- striped segments $\rightarrow$ valence; solid $\rightarrow$ sea quark and gluon



Nucleon axial coupling

- recent percent level determination of g_A

[C.Chang et al., Nature 558, 91 (2018); arXiv:1805.12130]



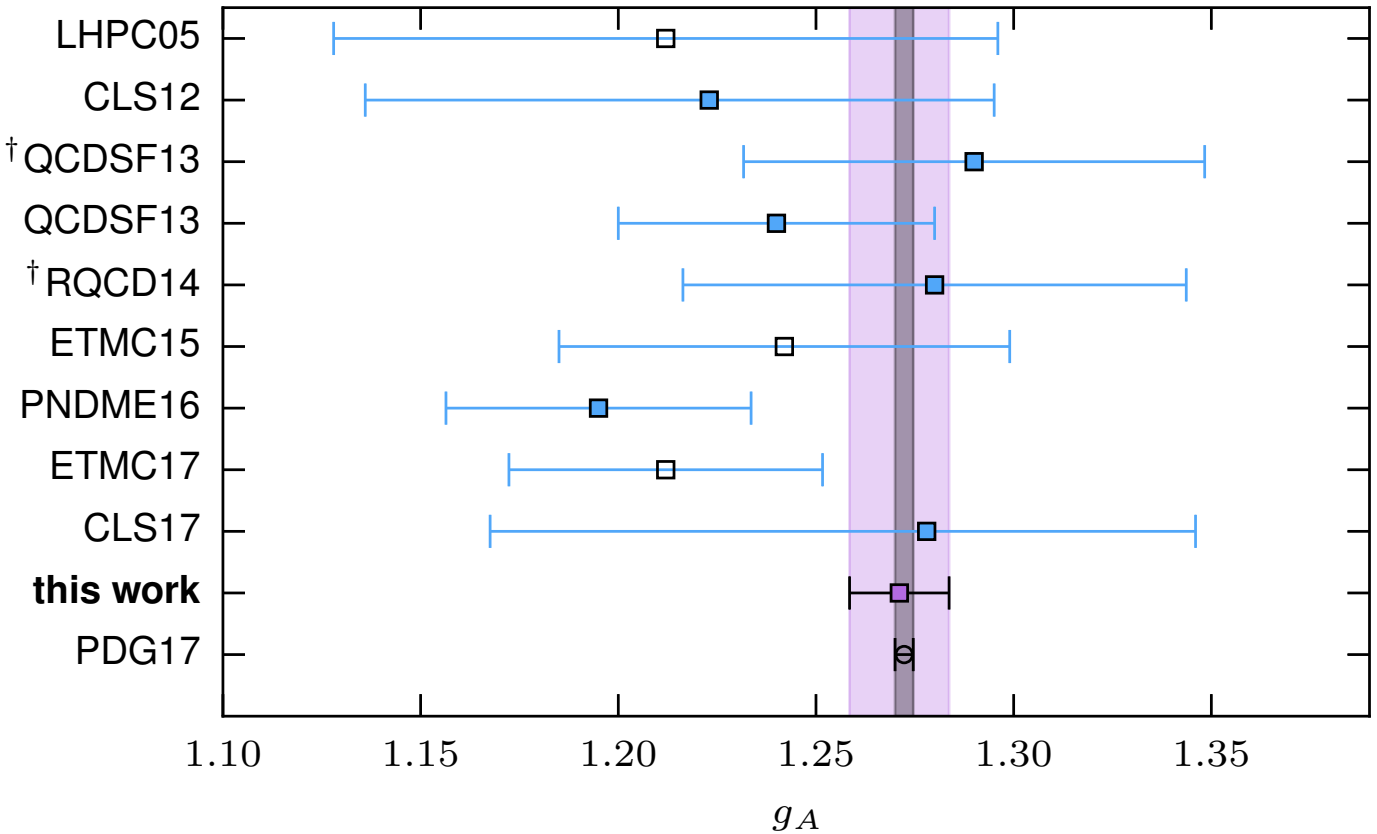
- use of Feynman-Hellman method

$$g_A = 1.2711(103)^s(39)^\chi(15)^a(19)^V(04)^I(55)^M$$

- errors: statistical,chiral,spacing,volume,isospin,model selection

Nucleon axial coupling (con't)

- comparison to other determinations

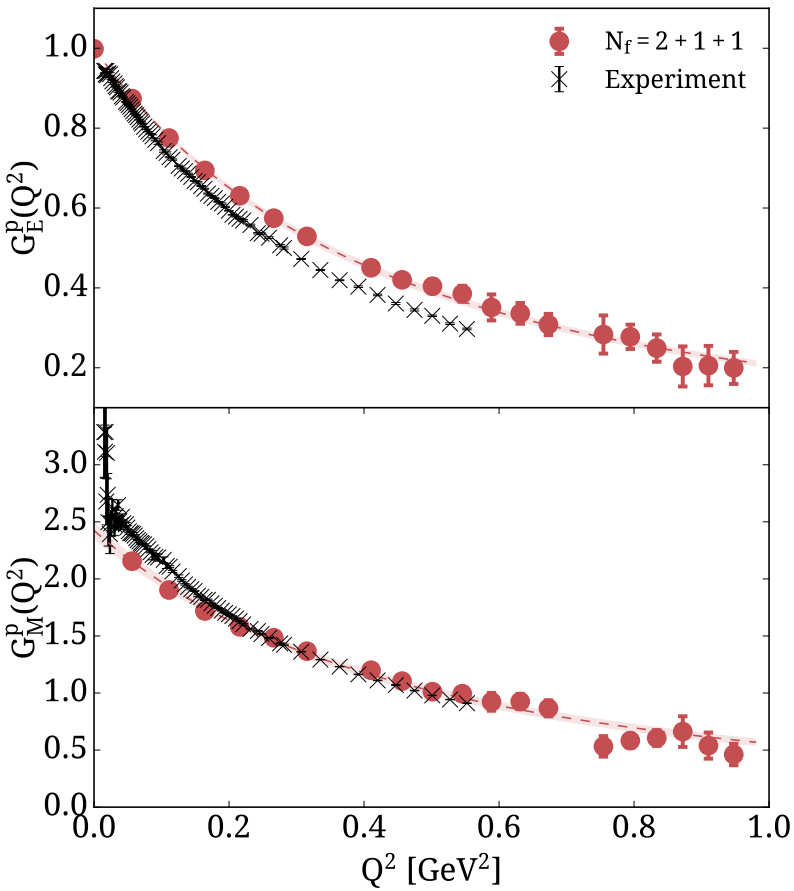


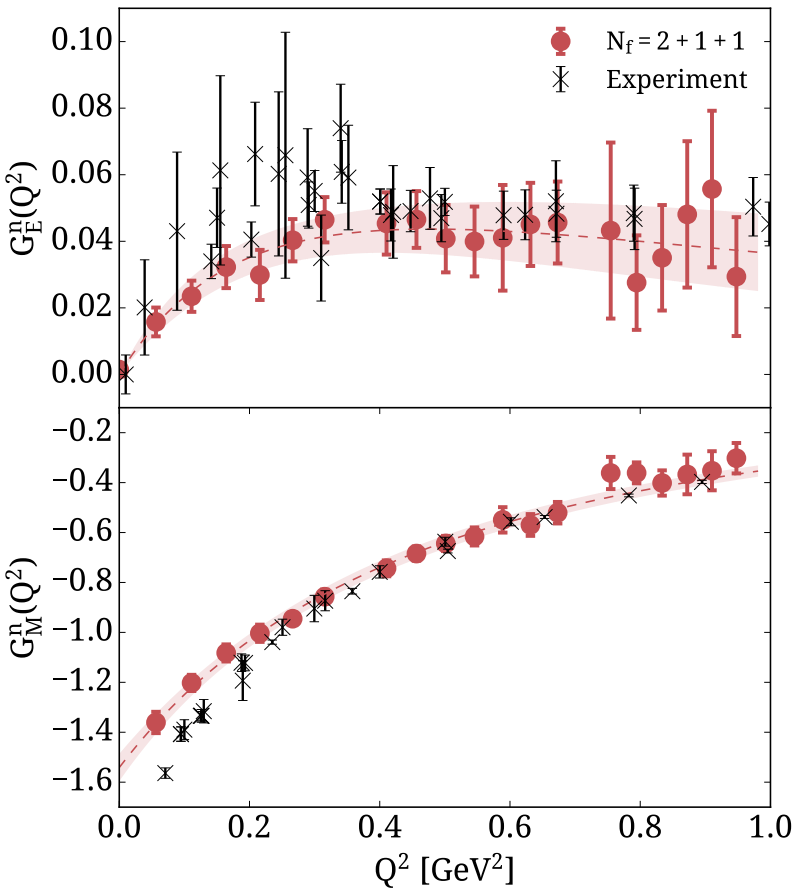
Proton/neutron electromagnetic form factors

- recent study of proton and neutron electromagnetic form factors
  [C.Alexandrou, S.Bacchio, M.Constantinou, J.Finkenrath, K.Hadjiyiannakou, K.Jansen, G.Koutsou, A.V.Aviles Casco, arXiv:1812.10311]
- one ensemble $N_f = 2 + 1 + 1$ twisted mass with $m_\pi = 130\ \mathrm{MeV}$
- two ensembles $N_f = 2$ twisted mass with $m_\pi = 130\ \mathrm{MeV}$ and two volumes $Lm_\pi \sim 3$ and $Lm_\pi \sim 4$
- unprecedented precision of disconnected diagram contributions
  - hierarchical probing
  - low mode deflation
  - large numbers of smeared point sources to reduce gauge noise
- disconnected diagrams have nonnegligible effects
- thorough investigation of excited-state contamination
- further study of finite-volume effects at low Q^2 needed

Proton/neutron electromagnetic form factors (con't)

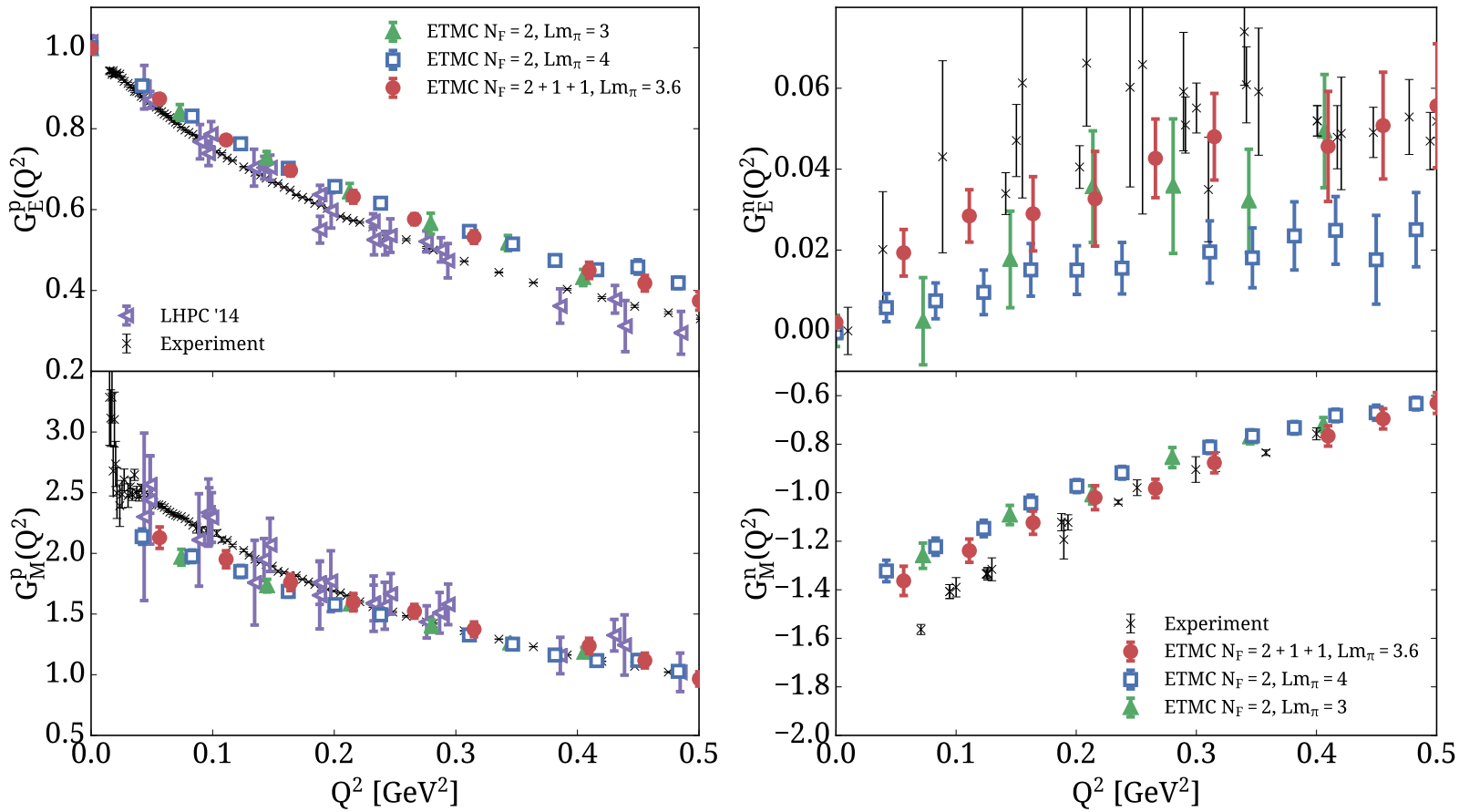
- comparison of $N_f = 2 + 1 + 1$ results to experiment





Proton/neutron electromagnetic form factors (con't)

- comparing $N_f = 2 + 1 + 1$ and $N_f = 2$ (hollow symbols ignore disconnected)

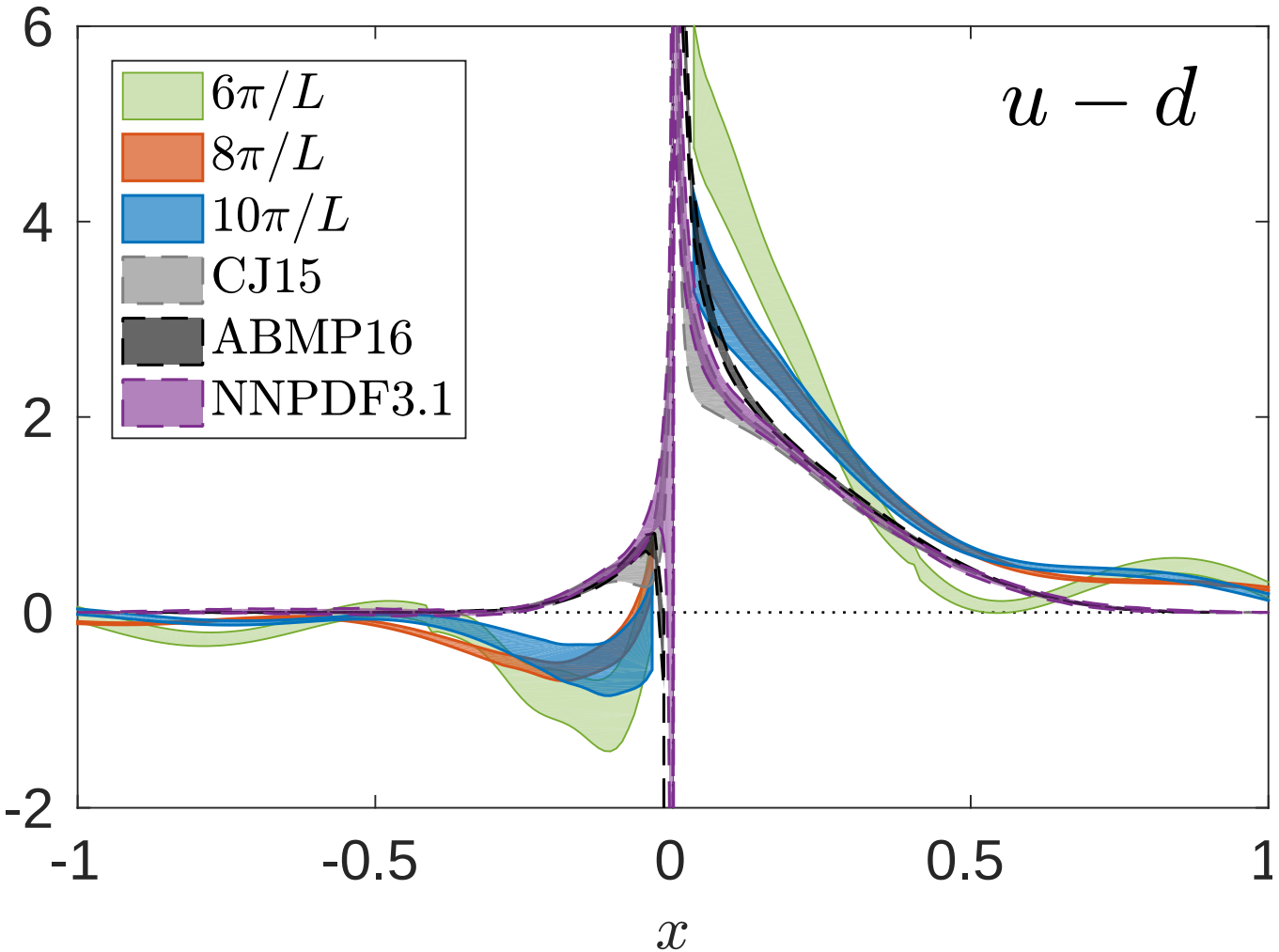


Light-cone parton distribution function

- first determination of unpolarized helicity parton distribution function at the physical point with nonperturbative renormalization and large momenta treated [C.Alexandrou, K.Cichy, M.Constantinou, K.Jansen, A.Scapellato, F.Steffens, PRL 121, 112001 (2018)]
- extracting PDFs from their moments impractical
- used method proposed by Ji [X.Ji, PRL110, 262002 (2013)] with subsequent refinements
  - compute spatial correlations between boosted nucleon states
  - Fourier transforms produce quasi-PDFs
  - take infinite-momentum limit via a refined matching procedure
  - target mass corrections
  - renormalization scheme for Wilson line operators
- one $48^3 \times 96$ twisted mass $N_f = 2$ ensemble $a = 0.0938(3)(2)$ fm and $m_\pi L = 2.98(1)$ at physical point

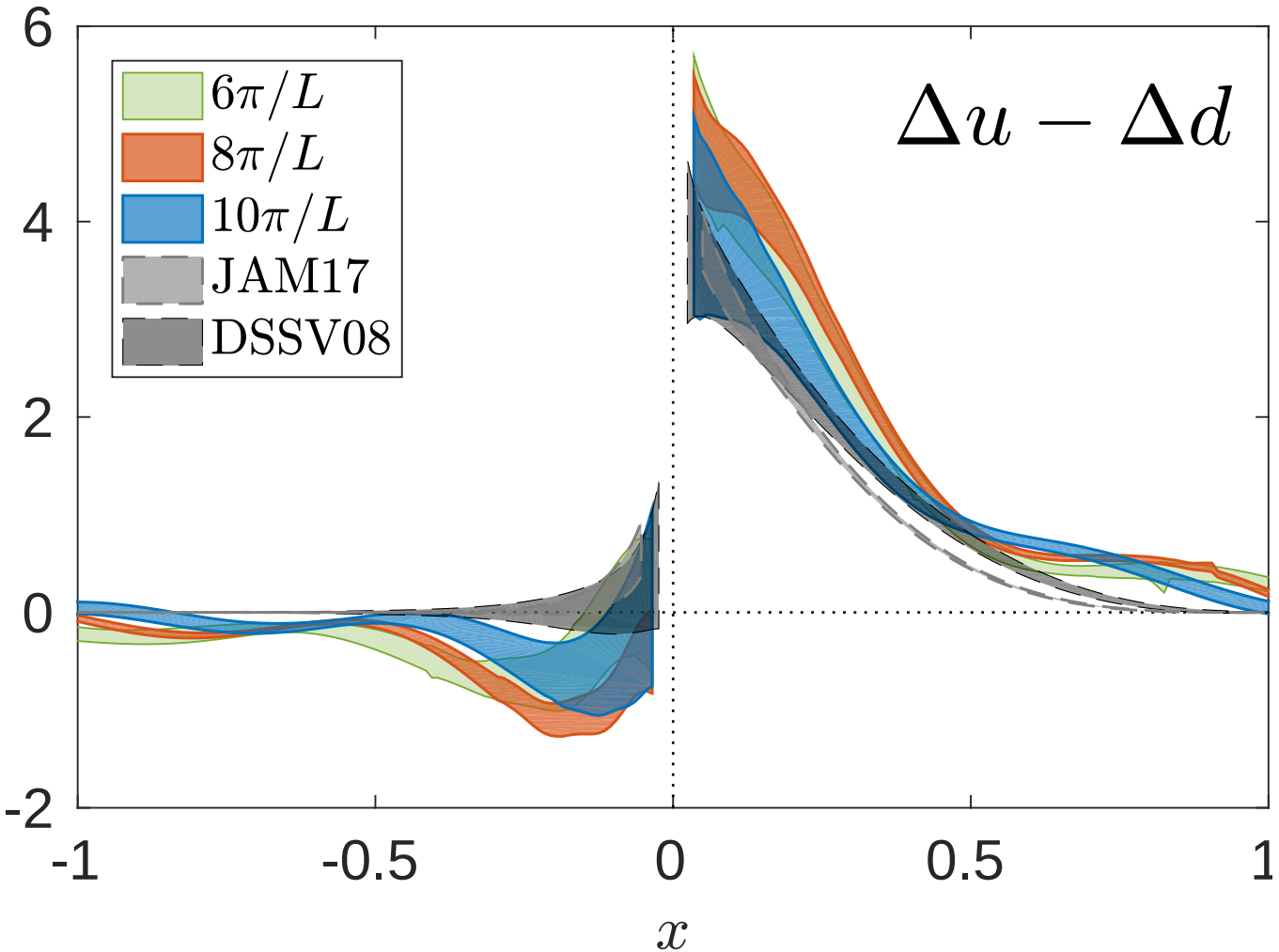
Light-cone parton distribution function (con't)

- unpolarized PDFs for three momenta compared to some phenomenological curves



Light-cone parton distribution function (con't)

- polarized PDFs for three momenta compared to some phenomenological curves



Conclusion

- stochastic LapH method works very well
  - allows evaluation of all needed quark-line diagrams
- large numbers of excited-state energy levels can be estimated
- scattering phase shifts can be computed
- infinite-volume resonance parameters from finite-volume energies below 2 particle thresholds
- time-like pion form factor computed
- scalar glueball determine in full QCD
- role of tetraquark operators
- string breaking revisited